

Character Formulae and Partition Functions in Higher Dimensional Conformal Field Theory

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A discussion of character formulae for positive energy unitary irreducible representations of the conformal group is given, employing Verma modules and Weyl group reflections. Product formulae for various conformal group representations are found. These include generalisations of those found by Flato and Fronsdal for $SO(3,2)$. In even dimensions the products for free representations split into two types depending on whether the dimension is divisible by four or not.

Keywords: Conformal field theory, Character formulae, Partition functions.

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1. Introduction

Motivated by the AdS/CFT correspondence, character formulae for groups associated with conformal symmetry have received greater attention recently [1,2,3,4]. In [1,2,4] the way in which character formulae encode the spectrum of operators allowed in a conformal Yang-Mills theory has been their main use. We hope that the present discussion might be similarly useful for conformal Yang-Mills theories in higher dimensions.

It is well known that character formulae provide an elegant way of decomposing tensor products of Lie algebra representations - the Racah-Speiser algorithm for decomposing tensor products of finite dimensional irreducible representations of simple Lie algebras may be easily proved in terms of Weyl characters, see [5] for a summary. In conformal field theories the method of characters was used by Flato and Fronsdal [6] to decompose products of certain massless representations, called ‘Di’ and ‘Rac’, in three dimensions. Oscillator and other methods have been used by various authors to generalise this to higher dimensions [7,8,9,10]. Here we follow a more direct approach using character formulae for the conformal group to decompose products of positive energy unitary irreducible representations of the conformal group which *inter alia* provides a generalisation of the Flato-Fronsdal results. These formulae may also be relevant to operator product expansions.

The layout of the paper is as follows. We rewrite the conformal algebra in terms of the orthonormal basis of $SO^*(d+2)$, the complexification of the conformal group in d dimensions, in section 2.

In section 3 we construct the characters of any positive energy unitary irreducible representation of the conformal group. The problem is related to finding characters of certain infinite dimensional representations of $SO^*(d+2)$ and we make use of a result in [11], employing Verma module characters, for solving it. The main part of the task consists, in this approach, of finding sub-Verma modules of an original one. This is more straightforward in the orthonormal basis of $SO^*(d+2)$ due to simplifications in the Weyl group action on weights in this basis. In this section we also show how these formulae are equivalent to ones obtained as follows. The basis for the original $SO^*(d+2)$ Verma module is reduced in a way determined by appendix C which discusses unitary representations of the conformal group. We write down the character of the reduced $SO^*(d+2)$ Verma module and then simply act on it with the Weyl symmetry operator of $SO(d)$. The formula obtained agrees with the character formula for the corresponding irreducible representation of $SO^*(d+2)$. We also give the three and four dimensional results explicitly and these match known results [4,6,12,13].

In section 4 we discuss products of the unitary irreducible representations. As a simple example we first discuss the case of $d=4$. This is made simpler by the fact that the $SO(4)$

character may be rewritten as a product of two $SO(3)$ characters. We then go on to discuss higher dimensional cases which correspond to products of free representations. Crucial in this approach are expansion formulae of the characters in the following form, namely,

$$\sum_N s^{\Delta+N} F_N(\underline{x}), \quad (1.1)$$

where Δ denotes the canonical conformal dimension, $F_N(\underline{x})$ is some linear combination of the $SO(d)$ characters and with s and $\underline{x} = (x_1, \dots, x_r)$, $r = [\frac{1}{2}d]$ being some variables. We give expansion formulae of the type (1.1) for all character formulae of interest.

In even dimensions the product formulae obtained divide into two forms depending on whether the dimension d is divisible by four or not.

While we do not find all such product formulae, we feel that the method presented generalises easily when used in conjunction with expansion formulae of the type (1.1).

Expansions of the form (1.1) are used in section 5 then to correlate our results for character formulae with one-particle partition functions which have been found by various authors [14,15,16] for the free scalar, Weyl fermion and $\frac{d}{2}$ -form field strengths. We also discuss an expansion formula given for the character for conserved symmetric traceless tensor currents in the main text. A simple argument is given which explains the behaviour of the character formula (in the form (1.1)) when $\underline{x} = (1, \dots, 1)$. Also it is found that the character for the free scalar obtained here matches the one particle partition function for a scalar field on the boundary of AdS in [17] when the spin of descendants is taken into account.

Various useful formulae and proofs are left in the remaining appendices. In appendix A results for character formulae for infinite dimensional representations of semi-simple Lie groups are discussed. In appendix B standard formulae for $SO(d)$ Weyl characters are given. In appendix C unitarity bounds are discussed. While unitarity bounds have been discussed in great detail by other authors, see for example [18,19,20], we feel that this attention is merited in that it determines which combination of generators are to be omitted from the full Verma module for $SO^*(d+2)$ when character formulae for unitary irreducible representations of the conformal group are analysed. Appendix D contains proofs of certain product and expansion formulae for conformal group characters.

2. The conformal algebra in the orthonormal basis

Starting from the Lie algebra for $SO(d, 2)$,

$$[M_{AB}, M_{CD}] = i(g_{AC} M_{BD} - g_{AD} M_{BC} - g_{BC} M_{AD} + g_{BD} M_{AC}), \quad (2.1)$$

for $A, B = 1, \dots, d+2$, $g_{AB} = \text{diag.}(1, \dots, 1, -1, -1)$ and where $M_{AB} = -M_{BA}$ are Hermitian, then (2.1) may be related to the standard form of the conformal algebra by defining M_{ab} , \mathcal{P}_a , \mathcal{K}_a , H , for $a, b = 1, \dots, d$ through

$$[M_{AB}] = \begin{pmatrix} M_{ab} & -M_{at} \\ M_{sb} & \epsilon_{st} H \end{pmatrix}, \quad (2.2)$$

where $s, t = d+1, d+2$, $\epsilon_{d+1, d+1} = 1$, $\epsilon_{st} = -\epsilon_{ts}$ and

$$\mathcal{P}_a = M_{a, d+1} + iM_{a, d+2}, \quad \mathcal{K}_a = M_{a, d+1} - iM_{a, d+2}. \quad (2.3)$$

Then

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\delta_{ac} M_{bd} - \delta_{ad} M_{bc} - \delta_{bc} M_{ad} + \delta_{bd} M_{ac}), \\ [M_{ab}, \mathcal{P}_c] &= i(\delta_{ac} \mathcal{P}_b - \delta_{bc} \mathcal{P}_a), \quad [M_{ab}, \mathcal{K}_c] = i(\delta_{ac} \mathcal{K}_b - \delta_{bc} \mathcal{K}_a), \\ [H, \mathcal{P}_a] &= \mathcal{P}_a, \quad [H, \mathcal{K}_a] = -\mathcal{K}_a, \quad [\mathcal{K}_a, \mathcal{P}_b] = -2iM_{ab} + 2\delta_{ab}H, \end{aligned} \quad (2.4)$$

with all other commutators not mentioned in (2.4) vanishing. As usual, $M_{ab} = -M_{ba}$ are Hermitian generators of $SO(d)$ rotations. Here H , the conformal Hamiltonian, is required to have a positive definite spectrum of eigenvalues for positive energy representations.

In (2.4) M_{ab} of course satisfy the Lie algebra of $SO(d)$. For what follows we will use the orthonormal basis for $SO(d)$ whereby the Cartan subalgebra is defined by

$$H_i = M_{2i-1, 2i}, \quad i = 1, \dots, r, \quad [H_i, H_j] = 0, \quad (2.5)$$

with raising and lowering operators formed from

$$E_{ij}^{\varepsilon\eta} = -E_{ji}^{\eta\varepsilon} = M_{2i-1, 2j-1} + i\varepsilon M_{2i, 2j-1} + i\eta M_{2i-1, 2j} - \varepsilon\eta M_{2i, 2j}, \quad i \neq j, \quad \varepsilon, \eta = \pm, \quad (2.6)$$

augmented by

$$E_i^\pm = M_{2i-1, 2r+1} \pm iM_{2i, 2r+1}, \quad (2.7)$$

for $SO(2r+1)$.

In $2r$ dimensions the commutation relations among the $SO(2r)$ generators in the orthonormal basis are given by

$$\begin{aligned} [H_i, E_{jk}^{\varepsilon\eta}] &= (\varepsilon\delta_{ij} + \eta\delta_{ik})E_{jk}^{\varepsilon\eta}, \\ [E_{ij}^{\varepsilon\eta}, E_{ij}^{\varepsilon'\eta'}] &= (\varepsilon - \varepsilon')(1 - \eta\eta')H_i + (\eta - \eta')(1 - \varepsilon\varepsilon')H_j, \\ [E_{ij}^{\varepsilon\eta}, E_{jk}^{\varepsilon'\eta'}] &= i(\varepsilon'\eta - 1)E_{ik}^{\varepsilon\eta'}, \quad i \neq k, \end{aligned} \quad (2.8)$$

where $i, j, k = 1, \dots, r$ and $\varepsilon, \varepsilon', \eta, \eta' = \pm$ with other commutators, that can not be obtained through the symmetry $E_{ij}^{\varepsilon\eta} = -E_{ji}^{\eta\varepsilon}$, vanishing.

Using standard orthogonal unit vectors $e_i \in \mathbb{R}^r$ then $E_{ij}^{+\pm}$ for $1 \leq i < j \leq r$ correspond to the set of positive roots $e_i \pm e_j$ while $E_{ij}^{-\pm}$ correspond to the set of negative roots $-e_i \pm e_j$. The simple roots $e_i - e_{i+1}$, $e_{r-1} + e_r$ for $1 \leq i \leq r-1$ correspond to the linearly independent set of raising operators E_{i+1}^{+-} , E_{r-1}^{++} . Similarly the linearly independent set of lowering operators is E_{i+1}^{-+} , $1 \leq i \leq r-1$, E_{r-1}^{-} .

In $2r+1$ dimensions we have additionally the following commutation relations involving the extra generators (2.7), namely,

$$\begin{aligned} [H_i, E_j^{\pm}] &= \pm \delta_{ij} E_j^{\pm}, \\ [E_i^{\varepsilon}, E_j^{\eta}] &= (\varepsilon - \eta) H_i, \quad [E_i^{\varepsilon}, E_j^{\eta}] = i E_{ij}^{\varepsilon\eta}, \quad i \neq j, \\ [E_{ij}^{\varepsilon\eta}, E_j^{\varepsilon'}] &= -[E_{ji}^{\eta\varepsilon}, E_j^{\varepsilon'}] = i(\varepsilon'\eta - 1) E_i^{\varepsilon}, \end{aligned} \quad (2.9)$$

for $i, j = 1, \dots, r$ with all other such vanishing.

For $SO(2r+1)$ E_i^{+} corresponds to the extra positive roots e_i while E_i^{-} corresponds to the extra negative roots $-e_i$. The simple roots $e_i - e_{i+1}$, $1 \leq i \leq r-1$ and e_r correspond to the linearly independent set of raising operators E_{i+1}^{+-} , E_r^{+} . The linearly independent set of lowering operators is E_{i+1}^{-+} , $1 \leq i \leq r-1$ along with E_r^{-} .

In the orthonormal basis of $SO(2r)$, the $2r$ dimensional vector representation has highest weight e_1 and all other weights in the weight system are given by $\pm e_i$, $1 \leq i \leq r$. The weight system may be represented diagrammatically by

$$\begin{array}{c} e_r \\ \begin{array}{c} E_{r-1}^{-+} \\ \nearrow \quad \searrow \\ e_{r-1} \end{array} \\ \begin{array}{c} E_{r-2}^{-+} \cdots e_{r-2} \xrightarrow{E_{r-2}^{-+}} e_{r-1} \end{array} \end{array} \quad \begin{array}{c} E_{r-1}^{-} \\ \nearrow \quad \searrow \\ -e_{r-1} \end{array} \quad \begin{array}{c} E_{r-2}^{-+} \cdots e_{r-2} \xrightarrow{E_{r-2}^{-+}} -e_{r-1} \end{array} \quad \begin{array}{c} E_{12}^{-+} \\ \nearrow \quad \searrow \\ -e_1 \end{array} \quad (2.10)$$

which also indicates the action of the lowering operators E_{i+1}^{-+} , $1 \leq i \leq r$, E_{r-1}^{-} . As a convenient basis for the \mathcal{P}_a , \mathcal{K}_a operators, which of course transform in the vector representation, we define

$$\mathcal{P}_{i\pm} = \mathcal{P}_{2i-1} \pm i\mathcal{P}_{2i}, \quad \mathcal{K}_{i\pm} = \mathcal{K}_{2i-1} \pm i\mathcal{K}_{2i}, \quad (2.11)$$

for which $[E_{i+1}^{+-}, \mathcal{P}_{1+}] = [E_{r-1}^{++}, \mathcal{P}_{1+}] = [E_{i+1}^{+-}, \mathcal{K}_{1+}] = [E_{r-1}^{++}, \mathcal{K}_{1+}] = 0$ and

$$[H_i, \mathcal{P}_{j\pm}] = \pm \delta_{ij} \mathcal{P}_{j\pm}, \quad [H_i, \mathcal{K}_{j\pm}] = \pm \delta_{ij} \mathcal{K}_{j\pm}, \quad (2.12)$$

so that $\mathcal{P}_{i\pm}$, $\mathcal{K}_{i\pm}$ correspond to the weights $\pm e_i$. In terms of explicit action of lowering operators, we have that

$$\begin{aligned} \mathcal{P}_{i\pm} &= -\frac{i}{2} [E_{1i}^{-\pm}, \mathcal{P}_{1+}], \quad i = 2, \dots, r-1, \quad \mathcal{P}_{1-} = \frac{1}{4} [E_{12}^{-+}, [E_{12}^{-}, \mathcal{P}_{1+}]] \\ \mathcal{K}_{i\pm} &= -\frac{i}{2} [E_{1i}^{-\pm}, \mathcal{K}_{1+}], \quad i = 2, \dots, r-1, \quad \mathcal{K}_{1-} = \frac{1}{4} [E_{12}^{-+}, [E_{12}^{-}, \mathcal{K}_{1+}]], \end{aligned} \quad (2.13)$$

which may be easily unravelled in terms of the action of the linearly independent set of lowering operators E_{i+1}^{-+} , $1 \leq i \leq r$, E_{r-1}^{-} .

In fact the entire conformal algebra in this basis may be written in terms of the orthonormal basis of the $SO^*(2r+2)$ algebra. Making the definitions

$$H_0 = -H, \quad E_{0i}^{-\pm} = -E_{i0}^{\pm-} = \mathcal{P}_{i\pm}, \quad E_{i0}^{\pm+} = -E_{0i}^{+\pm} = \mathcal{K}_{i\pm}, \quad i = 1, \dots, r, \quad (2.14)$$

(so that $-H$ is the extra Cartan subalgebra element and $\mathcal{K}_{i\pm}/\mathcal{P}_{i\pm}$ form the extra raising/lowering operators) then the conformal algebra may be shown to be equivalent to (2.8) for the range of the indices i, j, k being extended to $0, 1, \dots, r$. The linearly independent set of raising/lowering operators in this case is extended to include $\mathcal{K}_{1-}/\mathcal{P}_{1+}$.

The $2r+1$ dimensional vector representation of $SO(2r+1)$ has highest weight e_1 and the weight system is $\pm e_i$ and 0. Diagrammatically the weight system is given by

$$e_1 \xrightarrow{E_{12}^{-+}} e_2 \cdots e_{r-1} \xrightarrow{E_{r-1}^{-+}} e_r \xrightarrow{E_r^{-}} 0 \xrightarrow{E_r^{-}} -e_r \xrightarrow{E_{r-1}^{-+}} -e_{r-1} \cdots -e_2 \xrightarrow{E_{12}^{-+}} -e_1 \quad (2.15)$$

where we have indicated the action of lowering operators. In addition to (2.11) we may also choose

$$\mathcal{P}_0 = \mathcal{P}_{2r+1}, \quad \mathcal{K}_0 = \mathcal{K}_{2r+1}, \quad (2.16)$$

as extra elements of the basis for \mathcal{P}_a , \mathcal{K}_a operators and these commute with H_i , $1 \leq i \leq r$, so that they have weight 0.

Again, in this basis the conformal algebra may be written in terms of the orthonormal basis of $SO^*(2r+3)$. In addition to (2.14) in this case we define

$$E_0^{-} = \mathcal{P}_0, \quad -E_0^{+} = \mathcal{K}_0, \quad (2.17)$$

and along with (2.8) the extra commutation relations are given by (2.9) for the range of the indices i, j being extended to $0, 1, \dots, r$. Again, the linearly independent set of raising/lowering operators in this case is extended to include $\mathcal{K}_{1-}/\mathcal{P}_{1+}$.

In terms of the orthonormal basis, unitarity requires that

$$H^\dagger = H, \quad H_i^\dagger = H_i, \quad E_{ij}^{\varepsilon\eta\dagger} = E_{ij}^{-\varepsilon-\eta}, \quad E_i^{\varepsilon\dagger} = E_i^{-\varepsilon}, \quad \mathcal{P}_{i\varepsilon}^\dagger = \mathcal{K}_{i-\varepsilon}, \quad \mathcal{P}_0^\dagger = \mathcal{K}_0. \quad (2.18)$$

3. Character formulae for positive energy unitary irreducible representations

Essential in our approach to finding the character formulae for positive energy irreducible representations of the conformal group are $SO^*(d+2)$ Verma modules. These

have basis generated by the arbitrary action of lowering operators on the conformal highest weight state $|\Delta, \underline{\ell}\rangle^{\text{h.w.}}$ corresponding to the $SO^*(d+2)$ weight $\underline{\Lambda} = (-\Delta, \ell_1, \dots, \ell_r)$, $r = [\frac{1}{2}d]$ (where for $a = 0, 1, \dots, r$ then Λ_a are $H_a = -H, H_i$ eigenvalues). In what follows we assume that ℓ is a dominant integral highest weight with respect to $SO(d)$, so that in the orthonormal basis $\ell_i = \ell \cdot e_i \in \frac{1}{2}\mathbb{Z}$ and¹

$$\ell_1 \geq \dots \geq \ell_{r-1} \geq |\ell_r|, \quad (3.1)$$

for $SO(2r)$ while for $SO(2r+1)$

$$\ell_1 \geq \dots \geq \ell_r \geq 0, \quad (3.2)$$

these being the respective dominant Weyl chambers (or boundaries thereof).

The highest weight state $|\Delta, \underline{\ell}\rangle^{\text{h.w.}}$ satisfies

$$H_a |\Delta, \underline{\ell}\rangle^{\text{h.w.}} = \Lambda_a |\Delta, \underline{\ell}\rangle^{\text{h.w.}}, \quad \mathcal{K}_{1-} |\Delta, \underline{\ell}\rangle^{\text{h.w.}} = E_\alpha |\Delta, \underline{\ell}\rangle^{\text{h.w.}} = 0, \quad (3.3)$$

for α being the simple roots of $SO(d)$ so that

$$\{E_\alpha\} \rightarrow \{E_{ii+1}^{++}, 1 \leq i \leq r-1, E_{r-1r}^{++}\}, \quad (3.4)$$

for $SO(2r)$ while

$$\{E_\alpha\} \rightarrow \{E_{ii+1}^{++}, 1 \leq i \leq r-1, E_r^+\}, \quad (3.5)$$

for $SO(2r+1)$.

The Verma module \mathcal{V}_Λ with highest weight Λ therefore has basis

$$\prod_{\substack{v=i\varepsilon, 0 \\ 1 \leq i \leq r, \varepsilon=\pm}} \mathcal{P}_v^{n_v} \prod_{\alpha \in \Phi_-} E_\alpha^{n_\alpha} |\Delta, \underline{\ell}\rangle^{\text{h.w.}}, \quad (3.6)$$

for Φ_- denoting the set of negative roots of $SO(d)$ and with n_v, n_α all positive or zero integers, with $n_0 = 0$ for $SO(2r)$. As mentioned before, for $SO(2r)$ then $\{E_\alpha\} \rightarrow \{E_{ij}^{\pm}\}$ while for $SO(2r+1)$ then $\{E_\alpha\} \rightarrow \{E_{ij}^{\pm}, E_i^-\}$. Corresponding to the basis (3.6) the weights Λ' in the Verma module are given by

$$\Lambda'_0 = -\Delta - \sum_{\substack{v=i\varepsilon, 0 \\ 1 \leq i \leq r, \varepsilon=\pm}} n_v, \quad \ell' = \ell - \sum_{\alpha \in \Phi_-} n_\alpha \alpha + \sum_{i=1}^r (n_{i+} - n_{i-}) e_i. \quad (3.7)$$

¹ The Dynkin labels for even d are given by $\Lambda'_i = \ell_i - \ell_{i+1}$, $1 \leq i \leq r-2$ and $\Lambda'_{r-1} = \ell_{r-1} + \ell_r$, $\Lambda'_r = \ell_{r-1} - \ell_r$ while for odd d they are $\Lambda'_i = \ell_i - \ell_{i+1}$, $1 \leq i \leq r-1$ and $\Lambda'_r = 2\ell_r$ and these are the conditions for them to be non-negative integers.

In (3.6) we assume some fixed ordering of \mathcal{P}_v , E_α . This ordering may be arbitrarily chosen since if a different ordering is assumed then the resulting Verma module basis can be expressed in terms of that in (3.6) due to \mathcal{P}_v , E_α having commutators which are closed among themselves.

In appendix C we will use the form of the algebra in the last section, in terms of the orthonormal basis, to derive conditions necessary for conformal group representations to be unitary. These results are summarised by

$$\begin{aligned} \Delta &\geq \Delta_p = \ell_1 + d - p - 1, \quad p = 1, \dots, [\tfrac{1}{2}d] \quad \text{for} \quad \ell_1 = \ell_2 = \dots = \ell_p > \ell_{p+1}, \\ \Delta &\geq \tfrac{1}{2}d - 1 \quad \text{or} \quad \Delta = 0 \quad \text{for} \quad \underline{\ell} = 0, \end{aligned} \quad (3.8)$$

while for odd d we have in addition that

$$\Delta \geq [\tfrac{1}{2}d] \quad \text{for} \quad \ell_1 = \dots = \ell_{[\frac{1}{2}d]} = \tfrac{1}{2}. \quad (3.9)$$

It has been proven elsewhere that these conditions are sufficient [18] (in order for states in irreducible representations of the conformal group to have strictly positive norm). We impose these conditions on the representations we are interested in.

Along with the highest weight Λ the weight system for \mathcal{V}_Λ may contain other highest weights Λ^w being, for certain w in the relevant Weyl group \mathcal{W} , shifted (or affine) Weyl reflections given by,²

$$\Lambda^w = w(\Lambda + \rho) - \rho, \quad (3.10)$$

for ρ being the Weyl vector, $\rho = -\frac{1}{2} \sum_{\alpha \in \Phi_-} \alpha$. As described more in Appendix A, \mathcal{V}_{Λ^w} is a sub-Verma module if and only if the w can be made to satisfy condition (A.9). A necessary condition is that $\Lambda - \Lambda^w$ be expressible as a linear combination of positive roots with non-negative integer coefficients. This is equivalent to demanding that the state with weight Λ^w can be reached by applying lowering operators on the highest weight state with weight Λ . If the highest weight Λ is dominant integral then all \mathcal{V}_{Λ^w} are sub-Verma modules of \mathcal{V}_Λ .

As described in appendix A, in order to find the character of \mathcal{I}_Λ the first step is to find all Λ^w which are highest weights of sub-Verma modules. Below we give necessary conditions for $SO^*(d+2)$ weights to satisfy this for the highest weight having orthonormal

² A simple example is for the \mathcal{V}_ℓ Verma module of $SO(d)$ whereby the state $|\underline{\ell}^{\sigma_{12}}\rangle = (E_{12}^{-+})^{\ell_1 - \ell_2 + 1} |\underline{\ell}\rangle^{\text{h.w.}}$, for $\sigma_{12}(\ell_1, \ell_2, \dots) = (\ell_2, \ell_1, \dots)$, characterises a sub-Verma-module $\mathcal{V}_{\ell^{\sigma_{12}}}$, for $|\underline{\ell}\rangle^{\text{h.w.}}$ being annihilated by all $SO(d)$ raising operators. To see this it suffices to use that $[E_{12}^{+-}, (E_{12}^{-+})^n] = 4n(H_1 - H_2 + n - 1)(E_{12}^{-+})^{n-1}$. For unitary representations $E_{ij}^{\varepsilon\eta\dagger} = E_{ij}^{-\varepsilon-\eta}$ and $|\underline{\ell}^{\sigma_{12}}\rangle$ is null.

basis labels $\underline{\Lambda} = (-\Delta, \ell_1, \dots, \ell_r)$. We show how each such weight may be written as Λ'^w where $w \in \mathcal{W}_d$, the Weyl group of $SO(d)$, and $\Lambda' = (-\Delta', \underline{\ell}')$ has $\underline{\ell}'$ satisfying (3.1) or (3.2), so that ℓ' is a dominant integral weight of $SO(d)$, or else such that the weight ℓ' has a Dynkin label equal to -1 in which case such contributions vanish in χ_Λ . Using the results of appendix A we may then write the character as

$$\chi_\Lambda = \sum_{w \in \mathcal{W}_d} \text{sgn}(w) \mathcal{C}_{\Lambda^w} + \sum_{w \in \mathcal{W}_d, \Lambda'} \gamma_{\Lambda'} \text{sgn}(w) \mathcal{C}_{\Lambda'^w}, \quad (3.11)$$

where $\mathcal{C}_{\Lambda'}$ are $SO^*(d+2)$ Verma module characters and $\gamma_{\Lambda'}$ is determined by a recurrence relation. To solve this recurrence relation requires knowing in more detail which submodules are contained in which and so condition (A.9) applies here.

The Weyl group \mathcal{W}_{d+2} acts in a particularly simple way on weights of the Verma module \mathcal{V}_Λ of $SO^*(d+2)$ in the orthonormal basis. Choosing any $w \in \mathcal{W}_{d+2}$ then we may write

$$w(\Lambda_0, \dots, \Lambda_r) = (\varepsilon_0 \Lambda_{\sigma(0)}, \dots, \varepsilon_r \Lambda_{\sigma(r)}), \quad (3.12)$$

for $\sigma \in \mathcal{S}_{r+1}$ and $\varepsilon_a = \pm 1$ with $\prod_a \varepsilon_a = 1$ for $d = 2r$. In the present case the relevant Weyl vector has components

$$\rho_a = \frac{1}{2}d - a, \quad a = 0, \dots, r, \quad (3.13)$$

in the orthonormal basis of $SO^*(d+2)$. Notice that the last r components are the components of the Weyl vector for $SO(d)$. From (3.10) we have that the components of Λ^w in the orthonormal basis are

$$\Lambda_a^w = \varepsilon_a \Lambda_{\sigma(a)} + (\varepsilon_a - 1) \frac{1}{2}d - \varepsilon_a \sigma(a) + a. \quad (3.14)$$

Now for $SO^*(d+2)$ the weights $\underline{\Lambda} = (-\Delta, \ell_1, \dots, \ell_r)$ are clearly not dominant integral unless $\Delta = \ell_i = 0$ which corresponds to the trivial representation. Sub-Verma-module weights must satisfy (3.7). Thus, for any Λ^w to be the highest weight of a sub-Verma-module then $\Lambda_0^w = -\Delta - n$, $n \in \mathbb{N}$. Also the minimum value of Λ_0^w is that for $\varepsilon_0 = -1$, $\sigma(0) = 1$ so that sub-Verma-modules exist for

$$-\ell_1 - d + 1 \leq \Lambda_0^w = -\Delta - n. \quad (3.15)$$

Notice that for $\varepsilon_0 = 1$, $\sigma(0) = 0$ so that $\Lambda_0^w = -\Delta$ then all \mathcal{V}_{Λ^w} , $w \in \mathcal{W}_d$ are sub-Verma-modules as ℓ is a dominant integral highest weight with respect to the $SO(d)$ subgroup. For any other $\varepsilon_0, \sigma(0)$ then this corresponds to a definite action of \mathcal{P}_v on the highest weight state so that for these cases $n \geq 1$.

Using the formula (3.11) and the results of appendix A with condition (3.15) we now discuss the even and odd dimension character formulae separately.

Character formulae: even dimensions

For application of the results of appendix A we need to specify what condition (A.9) demands for $SO^*(d+2)$ Weyl group elements in $d = 2r$ dimensions. For S_{ab} , $T_{(ab)}$ being the \mathcal{W}_d element for the respective positive roots $e_a + e_b$, $e_a - e_b$, $0 \leq a < b \leq r$ then clearly

$$\begin{aligned} S_{ab}(\Lambda_0, \dots, \Lambda_a, \dots, \Lambda_b, \dots, \Lambda_r) &= (\Lambda_0, \dots, -\Lambda_b, \dots, -\Lambda_a, \dots, \Lambda_r), \\ T_{(ab)}(\Lambda_0, \dots, \Lambda_a, \dots, \Lambda_b, \dots, \Lambda_r) &= (\Lambda_0, \dots, \Lambda_b, \dots, \Lambda_a, \dots, \Lambda_r). \end{aligned} \quad (3.16)$$

$T_{(ab)}$ corresponds to the transposition (ab) and below we use the short-hand notation $T_\sigma = T_{(a_1 b_1)} \dots T_{(a_j b_j)}$ for $\sigma = (a_1 b_1) \dots (a_j b_j)$. Applying S_{ab} , respectively $T_{(ab)}$, to some weight $\underline{\Lambda}' = (\Lambda'_0, \dots, \Lambda'_r)$ then clearly condition (A.9) allows only those S_{ab} , respectively $T_{(ab)}$, for which $\Lambda'_a + \Lambda'_b \in \mathbb{N}$, respectively $\Lambda'_a - \Lambda'_b \in \mathbb{N}$.

We may easily write the character formula for the $SO^*(2r+2)$ Verma module with highest weight Λ , for $\underline{\Lambda} = (-\Delta, \ell)$, and weights Λ' given by (3.7) as

$$C_{\underline{\Lambda}}^{(2r+2)}(s, x) = \sum_{\Lambda'} e^{\Lambda'}(\mu) = s^\Delta C_{\underline{\ell}}^{(2r)}(x) P^{(2r)}(s, x), \quad (3.17)$$

where, for some general $SO^*(d+2)$ weight μ ,

$$s = e^{-e_0}(\mu) = e^{-\mu_0}, \quad x_i = e^{e_i}(\mu) = e^{\mu_i}, \quad (3.18)$$

$C_{\underline{\ell}}^{(2r)}(x)$ denotes the character of the $SO(2r)$ Verma module with highest weight ℓ (given in appendix B) and

$$P^{(2r)}(s, x) = \prod_{1 \leq i \leq r} (1 - s x_i)^{-1} (1 - s x_i^{-1})^{-1}. \quad (3.19)$$

($P^{(2r)}(s, x)$ comes from the summation over $n_{i\pm}$ implicit in (3.17).)

For $\Delta > \ell_1 + d - 2$ or Δ lying between two any of Δ_p and Δ_{p+1} in (3.8) then (3.15) implies that the only sub-Verma-modules are at most those having highest weights for $\varepsilon_0 = 1$, $\sigma(0) = 0$ in (3.14). However, since ℓ is a dominant integral weight of $SO(d)$ then all \mathcal{V}_{Λ^w} , $w \in \mathcal{W}_d$ are sub-Verma modules of \mathcal{V}_Λ in this case. Thus from (3.11) the corresponding character is,

$$\mathcal{A}_{[\Delta; \underline{\ell}]}^{(2r)}(s, x) = \sum_{w \in \mathcal{W}_{2r}} \text{sgn}(w) C_{\underline{\Lambda}^w}^{(2r+2)}(s, x) = s^\Delta \chi_{\underline{\ell}}^{(2r)}(x) P^{(2r)}(s, x). \quad (3.20)$$

Let us assume that $\ell_1 = \ell_2 = \dots = \ell_p > |\ell_{p+1}|$, $p \leq r-1$, $\Delta = \Delta_p$. In this case only Λ_0^w for $\varepsilon_0 = 1$, $\sigma(0) = 0$ and $\varepsilon_0 = -1$, $\sigma(0) = 1, \dots, p$ in (3.14) satisfy (3.15). For $\varepsilon_0 = 1$, $\sigma(0) = 0$ then all \mathcal{V}_{Λ^w} , $w \in \mathcal{W}_d$ are sub-Verma modules. Let us assume $\varepsilon_0 = -1$, $\sigma(0) = j$, $1 \leq j \leq p$ for which $\Lambda_0^w = -\ell_1 - d + j$ then it is not difficult to show that the rest of the components may be written as

$$(\ell_1, \dots, \ell_1, \ell_{p+1}, \dots)^w = (\ell_1, \dots, \ell_1, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ position}}}{\ell_1 - 1}, \dots, \ell_1 - 1, \ell_{p+1}, \dots)^{w'}, \quad w' \in \mathcal{W}_d, \quad (3.21)$$

whereby if in the original $w \in \mathcal{W}_{d+2}$ then $\sigma(k_a) = a$, $k_a \neq 0$, $a \neq j$ then w' is defined in terms of w by

$$\varepsilon'_0 = -\varepsilon_0 = 1, \quad \varepsilon'_{k_0} = -\varepsilon_{k_0}, \quad \varepsilon'_{k_i} = \varepsilon_{k_i}, \quad \sigma' = (0 \ p \ p-1 \dots j)\sigma, \quad (3.22)$$

so that $\sigma'(0) = 0$ and σ' exhausts all members of \mathcal{S}_r . Thus we have shown that all weights for the case of $w \in \mathcal{W}_{d+2}$ having $\varepsilon_0 = -1$, $\sigma(0) = j$ in (3.14) may be written as $\Lambda^{(j,p)w'}$, $w' \in \mathcal{W}_d$ with³

$$\underline{\Lambda}^{(j,p)} = (-\ell_1 - d + j, \ell_1, \dots, \ell_1, \underset{\substack{\uparrow \\ (j+1)^{\text{th}} \text{ position}}}{\ell_1 - 1}, \dots, \ell_1 - 1, \ell_{p+1}, \dots). \quad (3.23)$$

We may now easily show that $\Lambda^{(j,p)w'}$, $w' \in \mathcal{W}_d$ exhaust all other highest weights of sub-Verma-modules of \mathcal{V}_{Λ} in this case. To see this notice that if condition (A.9) of appendix A is satisfied for the weight $\Lambda^{(j,p)}$ then it is satisfied for $\Lambda^{(j,p)w'}$, $w' \in \mathcal{W}_d$ since the $(\ell_1, \dots, \ell_1, \ell_1 - 1, \dots, \ell_1 - 1, \ell_{p+1}, \dots)$ corresponds to a dominant integral weight of $SO(d)$. For $\Lambda^{(j,p)}$ itself we may show that $\Lambda^{(j,p)} = \Lambda^w$ where $w = S_{0p}T_{\sigma}$ for $\sigma = (p \ p-1 \dots j)$ satisfies condition (A.9). We see this as $(p \ p-1 \dots j) = (p \ p-1)(p-1 \ p-2) \dots (j+1 \ j)$ and $T_{(i+1 \ i)}T_{(i \ i-1)} \dots T_{(j+1 \ j)}(\underline{\Lambda} + \underline{\rho}) \cdot (\underline{e}_{i+2} - \underline{e}_{i+1}) \in \mathbb{N}$ for $i < p$ and $T_{\sigma}(\underline{\Lambda} + \underline{\rho}) \cdot (\underline{e}_0 + \underline{e}_p) \in \mathbb{N}$.

It now remains to determine $\gamma_{\Lambda^{(j,p)}} \rightarrow \gamma_{j,p}$ in (3.11) for the weight $\Lambda^{(j,p)}$ by the recurrence relation in appendix A. The first observation which is not difficult to show (in a

³ Also $\prod_a \varepsilon'_a = \prod_a \varepsilon_a$ and $\text{sgn}(w') = (-1)^{p+j+1} \text{sgn}(w)$. During the course of this work we noticed that, at this point, simply using the following formula

$$\chi_{\Lambda} = \mathcal{C}_{\Lambda} + \sum_{\substack{w \in \mathcal{W}, w \neq 1 \\ \Lambda^w \prec \Lambda}} \text{sgn}(w) \mathcal{C}_{\Lambda^w}$$

where the sum runs over all w satisfying condition (A.9), gives exactly the same result for the character as we find here by a more laborious procedure. It would be interesting to know whether or not this formula holds for more general Lie algebras and highest weights. We have not been able to find such a simple formula in the literature.

similar fashion as above) is that among $\mathcal{V}_{\Lambda^{(k,p)}}$ the only such modules which contain $\mathcal{V}_{\Lambda^{(j,p)}}$ as a sub-module are those for $j < k \leq p$ or none such if $j = p$. In fact it is possible to show that no proper sub-module of \mathcal{V}_{Λ} contains $\mathcal{V}_{\Lambda^{(p,p)}}$ so that in this case $\gamma_{p,p} = -1$.

To describe which sub-modules contain other $\mathcal{V}_{\Lambda^{(j,p)}}$ we first define a subset of the permutation group $\mathcal{T}_n \subset \mathcal{S}_n$ so that every $\tau \in \mathcal{T}_n$ has $1 \leq c \leq n$ cycles, where the first cycle consists of the first n_1 of the integers $n, n-1, \dots, 2, 1$, preserving this ordering, the second consists of the next n_2 such integers, again preserving this ordering, and so on and where $n_1, \dots, n_c \geq 1$ satisfy $\sum_{i=1}^c n_i = n$. For example for $n = 3$ then $\mathcal{T}_n = \{(321), (32)(1) = (32), (3)(21) = (21), (3)(2)(1) = 1\}$. It is not difficult to see that the number of such permutations with c cycles (counting trivial one cycles) is $\binom{n-1}{c-1}$ so that the total number of such permutations is 2^{n-1} . Further we have that for $n > 1$ there are 2^{n-2} of these permutations with signature 1 or -1 so that $\sum_{\tau \in \mathcal{T}_n} \text{sgn}(\tau) = 0$.

With the above definition of \mathcal{T}_n we have found that the sub-modules $\mathcal{V}_{\Lambda^{w'}}, w' \in \mathcal{W}_d, w' \neq 1$ contain $\mathcal{V}_{\Lambda^{(j,p)}}$ as a sub-module only for $w' \in \mathcal{T}_{p+1-j}$ so that for $j > p$ then $\sum_{w'} \text{sgn}(w') = -1$ is the contribution to $\gamma_{j,p}$ coming from these. Also we have found that the sub-modules $\mathcal{V}_{\Lambda^{(k,p)w'}}, j < k \leq p, w' \in \mathcal{W}_d$ containing $\mathcal{V}_{\Lambda^{(j,p)}}$ have $w' \in \mathcal{T}_{k-j}$ so that for $j+1 < k \leq p$ then $\sum_{w'} \text{sgn}(w') = 0$ so that the contribution to $\gamma_{j,p}$ coming from these vanishes while for $k = j+1$ then $w' = 1$ and this contributes $\gamma_{j+1,p} \text{sgn}(w') = \gamma_{j+1,p}$ to $\gamma_{j,p}$. With these results we may then easily find that

$$\gamma_{j,p} = (-1)^{p+j+1}, \quad (3.24)$$

solves the recurrence relation $\gamma_{j,p} = -\gamma_{j+1,p}, \gamma_{p,p} = -1$.

It is possible to show that for such w' as described above (essentially belonging to \mathcal{T}_n for various n) then for $w \in \mathcal{W}_{d+2}$ so that $\Lambda^{(j,p)} = \Lambda^{ww'}$ or $\Lambda^{(j,p)} = \Lambda^{(k,p)ww'}$ the only w which satisfy condition (A.9) are expressible as products of $S_{0l}, T_{(mn)}, 1 \leq l, m, n \leq p$.

Now applying (3.11) and the Weyl character formula for $SO(2r)$ in terms of (3.17) (appendix B) we find that the corresponding character is,

$$\begin{aligned} \mathcal{D}_{[\ell_1+2r-p-1; \ell_1, \ell_{p+1}, \dots, \ell_r]}^{(2r)}(s, x) \\ = s^{\ell_1+2r-p-1} \left(\chi_{\underline{\ell}}^{(2r)}(x) + \sum_{1 \leq j \leq p} (-s)^{p+1-j} \chi_{\underline{\ell} - \underline{e}_p - \dots - \underline{e}_j}^{(2r)}(x) \right) P^{(2r)}(s, x). \end{aligned} \quad (3.25)$$

for $\underline{\ell} = (\ell_1, \dots, \ell_1, \ell_{p+1}, \dots, \ell_r)$.

Notice that for even dimensions we also have the possibility of $\ell_1 = \dots = \ell_{r-1} = \pm \ell_r, \Delta = \ell_1 + \frac{1}{2}d - 1$. Here the Λ_0^w satisfying (3.15) are those for $\varepsilon_0 = 1, \sigma(0) = 0, \varepsilon_0 = \mp 1, \sigma(0) = r$ along with $\varepsilon_0 = -1, \sigma(0) = 1, \dots, r-1$. By an argument very similar

to the previous we find that, for $\underline{\ell} = (\ell, \dots, \ell, \pm\ell)$,

$$\begin{aligned} \mathcal{D}_{[\ell+r-1; \underline{\ell}] \pm}^{(2r)}(s, x) \\ = s^{\ell+r-1} \left(\chi_{\underline{\ell}}^{(2r)}(x) + \sum_{1 \leq j \leq r} (-s)^{r+1-j} \chi_{\underline{\ell} \mp \underline{e}_r - \dots - \underline{e}_j}^{(2r)}(x) \right) P^{(2r)}(s, x), \end{aligned} \quad (3.26)$$

is the corresponding character in this case.

The free scalar case, for which $\underline{\Lambda} = (-r+1, 0, \dots, 0)$ is the highest $SO(2r+2)$ weight, is accounted for by (3.26) for $\ell = 0$ and has character

$$\mathcal{D}_{[r-1; 0]}^{(2r)}(s, x) \equiv \mathcal{D}_{[r-1; 0] \pm}^{(2r)}(s, x) = s^{r-1} (1 - s^2) P^{(2r)}(s, x), \quad (3.27)$$

since the $SO(2r)$ characters obey $\chi_{(0, \dots, 0, -1, \pm 1)}(x) = -1$ with $\chi_{(0, \dots, 0, \pm 1)}(x) = 0$ and $\chi_{(0, \dots, 0, -1, \dots, -1, \pm 1)}(x) = 0$ otherwise.

Character formulae: odd dimensions

Considerations for odd $d = 2r + 1$ dimensions are very similar to those above for even $d = 2r$ dimensions and we will not go into as much detail here. Along with (3.16) \mathcal{W}_{2r+1} has extra elements corresponding to the extra positive roots e_i , $1 \leq i \leq r$ given by

$$S_a(\Lambda_0, \dots, \Lambda_a, \dots, \Lambda_r) = (\Lambda_0, \dots, -\Lambda_a, \dots, \Lambda_r). \quad (3.28)$$

Acting on some weight $\underline{\Lambda}' = (\Lambda'_0, \dots, \Lambda'_r)$ then condition (A.9) only allows those S_a for which $\Lambda'_a \in \frac{1}{2}\mathbb{N}$.

We may easily write the character formula for the $SO^*(2r+3)$ Verma module with highest weight Λ and weights Λ' given by (3.7) as

$$C_{\underline{\Lambda}}^{(2r+3)}(s, x) = \sum_{\Lambda'} e^{\Lambda'}(\mu) = s^{\Delta} C_{\underline{\ell}}^{(2r+1)}(x) P^{(2r+1)}(s, x), \quad (3.29)$$

where $C_{\underline{\ell}}^{(2r+1)}(x)$ denotes the character of the $SO(2r+1)$ Verma module with highest weight ℓ (given in appendix B) and

$$P^{(2r+1)}(s, x) = (1 - s)^{-1} \prod_{1 \leq i \leq r} (1 - sx_i)^{-1} (1 - sx_i^{-1})^{-1}. \quad (3.30)$$

($P^{(2r+1)}(s, x)$ comes from the summation over $n_{i\pm}, n_0$ implicit in (3.29).)

For $\Delta > \ell_1 + d - 2$ or Δ lying between two any of Δ_p and Δ_{p+1} in (3.8) then character for positive energy unitary irreducible representations is,

$$\mathcal{A}_{[\Delta; \underline{\ell}]}^{(2r+1)}(s, x) = \sum_{w \in \mathcal{W}_{2r+1}} \text{sgn}(w) C_{\underline{\Lambda}^w}^{(2r+3)}(s, x) = s^{\Delta} \chi_{\underline{\ell}}^{(2r+1)}(x) P^{(2r+1)}(s, x), \quad (3.31)$$

where $\chi_{\underline{\ell}}^{(2r+1)}(\mathbf{x})$ is the character of the $SO(2r+1)$ irreducible representation with highest weight $\underline{\ell}$.

For $\Delta = \Delta_p$ in (3.8) we may go through the same procedure as for the even dimensional case and find that the extra Weyl reflections (3.28) lead to nothing new as far as condition (A.9) is concerned. Thus for odd d and with $\underline{\ell} = (\ell_1, \dots, \ell_1, \ell_{p+1}, \dots, \ell_r)$ the corresponding character for these representations is,

$$\begin{aligned} \mathcal{D}_{[\ell_1+2r-p; \ell_1, \ell_{p+1}, \dots, \ell_r]}^{(2r+1)}(s, \mathbf{x}) \\ = s^{\ell_1+2r-p} \left(\chi_{\underline{\ell}}^{(2r+1)}(\mathbf{x}) + \sum_{1 \leq j \leq p} (-s)^{p+1-j} \chi_{\underline{\ell} - \underline{e}_p - \dots - \underline{e}_j}^{(2r+1)}(\mathbf{x}) \right) P^{(2r+1)}(s, \mathbf{x}). \end{aligned} \quad (3.32)$$

For the free scalar case, for which $\underline{\Lambda} = (-r + \frac{1}{2}, 0, \dots, 0)$ is the highest $SO(2r+3)$ weight, we have that the corresponding character is given by,

$$\mathcal{D}_{[r-\frac{1}{2}; 0]}^{(2r+1)}(s, \mathbf{x}) = s^{r-\frac{1}{2}} (1 - s^2) P^{(2r+1)}(s, \mathbf{x}). \quad (3.33)$$

For odd dimensions we also have the possibility of the highest weight Λ having components $\underline{\Lambda} = (-r, \frac{1}{2}, \dots, \frac{1}{2})$. This time the Λ_0^w satisfying (3.15) are those for $\varepsilon_0 = \pm 1$, $\sigma(0) = 0$, $\varepsilon_0 = -1$, $\sigma(0) = j$, $1 \leq j \leq r$. For $\varepsilon_0 = -1$, $\sigma(0) = 0$ then $\Lambda_0^w = -r - 1$ and the remaining components may be rewritten as $\underline{\ell}^w = \underline{\ell}^{w'}$ where $w' \in \mathcal{W}_{2r+1}$ is identical to w save for $\varepsilon'_0 = -\varepsilon_0 = 1$ so that $\prod_a \varepsilon'_a = -\prod_a \varepsilon_a$ and $\text{sgn}(w') = -\text{sgn}(w)$. The cases of $\varepsilon_0 = -1$, $\sigma(0) = j$, $1 \leq j \leq r$ are accounted for similarly as for even dimensions for $p = r$. However these cases have highest weights which are shifted \mathcal{W}_{2r+1} Weyl group reflections of $(-\Delta', \underline{\ell}') = (-2r - 1 + j, -\frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, \dots)$. For these weights at least one of the Dynkin labels $\ell'_{i+1} - \ell'_i$ is equal to -1 so that contributions from all these Verma modules vanish from the character formula (by a result of appendix A).

The only cases we have to consider are for the Verma modules $\mathcal{V}_{\Lambda' w'}$, $w' \in \mathcal{W}_{2r+1}$ for $\underline{\Lambda}' = (-r - 1, \frac{1}{2}, \dots, \frac{1}{2})$. By similar arguments as before all $\mathcal{V}_{\Lambda' w'}$ for $w' \neq 1$ are sub-modules of $\mathcal{V}_{\Lambda'}$. Due to $\underline{\Lambda}' + \underline{\rho} = S_0(\underline{\Lambda} + \underline{\rho})$, where S_a is defined in (3.28), then condition (A.9) (which is satisfied due to $\mathbf{e}_0 \cdot (\underline{\Lambda} + \underline{\rho}) = \frac{1}{2}$) implies that all $\mathcal{V}_{\Lambda' w'}$, $w' \in \mathcal{W}_{2r+1}$ are sub-modules of \mathcal{V}_{Λ} . Further we may show that $\mathcal{V}_{\Lambda'}$ is contained only in \mathcal{V}_{Λ} , and no sub-modules of \mathcal{V}_{Λ} , and so $\gamma_{\Lambda'} = -1$.

Taking into account these considerations, we have that the character is, from (3.11),

$$\begin{aligned} \mathcal{D}_{[r; \frac{1}{2}]}^{(2r+1)}(s, \mathbf{x}) &= \sum_{w \in \mathcal{W}_{2r+1}} \text{sgn}(w) (\mathcal{C}_{\Lambda^w} - \mathcal{C}_{\Lambda'^w}) \\ &= s^r (x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}}) \dots (x_r^{\frac{1}{2}} + x_r^{-\frac{1}{2}}) (1 - s) P^{(2r+1)}(s, \mathbf{x}), \end{aligned} \quad (3.34)$$

and where we have used that

$$\chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(2r+1)}(x) = (x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}}) \dots (x_r^{\frac{1}{2}} + x_r^{-\frac{1}{2}}). \quad (3.35)$$

Relation with reduced Verma module bases

We wish to show here that omitting certain of \mathcal{P}_v from the original Verma module basis (3.6) leads to formulae for characters which are equivalent to those obtained in the last sections.

Descendant states (i.e. those obtained by definite action of \mathcal{P}_v on the highest weight state) are $SO(d)$ representations belonging to the decomposition of $\mathbf{e}_1 \otimes \dots \otimes \mathbf{e}_1 \otimes \ell$ in terms of irreducible representations. From appendix C, these states are null for either of two reasons. One reason is that ℓ may lie on the boundary of the dominant Weyl chamber (3.1) (i.e. that some $\ell_1 = \ell_2 = \dots$) so that certain descendant states are null with respect to the $SO(d)$ subgroup. The other reason is that Δ may lie on a unitarity bound. By omitting the correct \mathcal{P}_v from (3.6) we effectively discard states in the original Verma module which are null due to the value of Δ . Acting with the Weyl symmetry operator on the character of the reduced Verma module is equivalent to projecting out of the reduced module states which are null by virtue of which $SO(d)$ representation they belong to. We may show that this prescription gives the same formulae for characters as found earlier.

For definiteness we consider the case where $\ell_1 = \dots = \ell_p > |\ell_{p+1}|$, $\Delta \geq \ell_1 + d - p - 1$, in even dimensions, $d = 2r$, although the other cases of interest are similar.

For this case and in the notation of appendix C, consider the $SO(2r)$ highest weight state $|\Delta + 1, \underline{\ell} - \underline{\mathbf{e}}_p\rangle$. The construction of such a state is non-trivial and in appendix C only the simplest such states have been constructed. Nevertheless we may write down the state in principle as

$$|\Delta + 1, \underline{\ell} - \underline{\mathbf{e}}_p\rangle = \mathcal{A}_{\underline{\ell}} \mathcal{P}_{p-} |\Delta, \underline{\ell}\rangle^{\text{h.w.}} + \sum_{v', \underline{\ell}'} \mathcal{B}_{v', \underline{\ell}'} \mathcal{P}_{v'} |\Delta, \underline{\ell}'\rangle, \quad (3.36)$$

where $\mathcal{P}_{v'} \leftrightarrow v' \in \{\pm \mathbf{e}_i\}$ is the subset of $\mathcal{P}_{i\pm}$ which may be reached by applying $SO(2r)$ raising operators to \mathcal{P}_{p-} and $\underline{\ell}' + \underline{\mathbf{v}}' = \underline{\ell} - \underline{\mathbf{e}}_p$. The subset $\mathcal{P}_{v'}$ may be easily determined from (2.10) to be given by $\mathcal{P}_{i+}, \mathcal{P}_{j-}$ for $1 \leq i \leq r$ and $p+1 \leq j \leq r$. The complex numbers $\mathcal{A}_{\underline{\ell}}, \mathcal{B}_{v', \underline{\ell}'}$ are determined by the condition that (3.36) be a highest weight state with respect to $SO(2r)$ i.e. that all $SO(2r)$ raising operators annihilate it. By the results of appendix C, for Δ above the unitarity bound this state is not null however when $\Delta = \ell_1 + d - p - 1$ then $|\Delta + 1, \underline{\ell} - \underline{\mathbf{e}}_p\rangle = 0$. This is equivalent to a conservation equation for the highest weight state $|\Delta, \underline{\ell}\rangle^{\text{h.w.}}$.

Now the modulus of $\mathcal{A}_{\underline{\ell}}$ is non-zero despite $\ell_1 = \dots = \ell_p > |\ell_{p+1}|$. When $\Delta = \ell_1 + d - p - 1$, so that (3.36) vanishes, then $\mathcal{P}_{p-}|\Delta, \underline{\ell}\rangle^{\text{h.w.}}$ may be expressed in terms of $\mathcal{P}_{i+}|\Delta, \ell'\rangle$, $\mathcal{P}_{j-}|\Delta, \ell'\rangle$ for $1 \leq i \leq r$ and $p+1 \leq j \leq r$. Also, as

$$E_{i+1}^{-+}|\Delta, \ell_1, \dots, \ell_1, \ell_{p+1}, \dots, \ell_r\rangle^{\text{h.w.}} = 0, \quad i = 1, \dots, p-1, \quad (3.37)$$

then applying such E_{i+1}^{-+} to (3.36) and using (2.10), we have that for Δ on the unitarity bound then $\mathcal{P}_{i-}|\Delta, \underline{\ell}\rangle^{\text{h.w.}}$, $1 \leq i \leq p-1$ may be similarly expressed in terms of $\mathcal{P}_{i+}|\Delta, \ell'\rangle$, $\mathcal{P}_{j-}|\Delta, \ell'\rangle$ for $1 \leq i \leq r$ and $p+1 \leq j \leq r$. Thus, effectively the Verma module basis (3.6) becomes reduced so as to exclude \mathcal{P}_{i-} , $1 \leq i \leq p$.

Acting with the $SO(d)$ Weyl symmetry operator \mathfrak{W}_d (see appendices A,B) on the character for the reduced Verma module yields the following formula,

$$\begin{aligned} \sum_{\Lambda', w \in \mathcal{W}_{2r}} e^{w(\Lambda')}(\mu) &= s^{\ell_1+2r-p-1} \mathfrak{W}_d \left(C_{\underline{\ell}}^{(2r)}(x) (1 - sx_1^{-1}) \dots (1 - sx_p^{-1}) \right) P^{(2r)}(s, x) \\ &= s^{\ell_1+2r-p-1} \left(\chi_{\underline{\ell}}^{(2r)}(x) + \sum_{\substack{1 \leq n \leq p \\ i_j \in \{1, \dots, p\}, i_j \neq i_k}} (-s)^n \chi_{\underline{\ell} - \mathbf{e}_{i_1} - \dots - \mathbf{e}_{i_n}}^{(2r)}(x) \right) P^{(2r)}(s, x), \end{aligned} \quad (3.38)$$

for $\underline{\ell} = (\ell_1, \dots, \ell_1, \ell_{p+1}, \dots, \ell_r)$ and where now $\underline{\Lambda}' = (-\Lambda'_0, \underline{\ell}')$ are specified by

$$\begin{aligned} \Lambda'_0 = -\Delta' &= -\ell_1 - d + p + 1 - \sum_{1 \leq i \leq r} n_{i+} - \sum_{p+1 \leq j \leq r} n_{j-}, \\ \ell' &= \ell - \sum_{\alpha \in \Phi_-} \alpha + \sum_{1 \leq i \leq r} n_{i+} \mathbf{e}_i - \sum_{p+1 \leq j \leq r} n_{j-} \mathbf{e}_j. \end{aligned} \quad (3.39)$$

It is easy to see that (3.38) reduces to (3.25). To see this note that $\chi_{\underline{\ell}'}(x)$ in (3.38) is non-zero only for $\underline{\ell}' = (\ell_1, \dots, \ell_1, \ell_1 - 1, \dots, \ell_1 - 1, \ell_{p+1}, \dots, \ell_r)$ i.e. for $i_j = k, k+1, \dots, p$ for some $1 \leq k \leq p$.

To summarise: when the conformal dimension Δ saturates a unitarity bound the Verma module basis is reduced so as to exclude certain of the $\mathcal{P}_{i\pm}$, \mathcal{P}_0 from (3.6). This is equivalent to conservation equations constraining the highest weight state. This subset may be determined in terms of the results of appendix C. Acting with the $SO(d)$ Weyl symmetry operator on the character of the reduced Verma module then leads to the characters for the corresponding unitary irreducible representation. Explicitly, for (3.25) or (3.32) the subset to be omitted from (3.6) is \mathcal{P}_{i-} , $1 \leq i \leq p$, for (3.26) the subset is \mathcal{P}_{i-} , $1 \leq i \leq r-1$ along with \mathcal{P}_{r-} for $\mathcal{D}_{[\ell+r-1; \ell]+}^{(2r)}$ or \mathcal{P}_{r+} for $\mathcal{D}_{[\ell+r-1; \ell]-}^{(2r)}$ while for (3.34) the subset is \mathcal{P}_{i-} , $1 \leq i \leq r$ along with \mathcal{P}_0 .

Special cases

We here illustrate these character formulae for the simplest cases of the $SO(3, 2)$ and $SO(4, 2)$ conformal groups and mention how special cases relate to conformal field representations.

We have that,

$$C_\ell(x) \equiv C_\ell^{(3)}(x) = \frac{x^\ell}{1 - x^{-1}}, \quad (3.40)$$

is the character of $SO(3)$ Verma modules and

$$\chi_\ell(x) \equiv \chi_\ell^{(3)}(x) = C_\ell(x) + C_\ell(x^{-1}) = \frac{x^{\ell+\frac{1}{2}} - x^{-\ell-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}, \quad (3.41)$$

is the usual character for $SO(3)$ irreducible representations. We have therefore that,⁴

$$\begin{aligned} \mathcal{A}_{[\Delta;\ell]}^{(3)}(s, x) &= s^\Delta \chi_\ell(x) P^{(3)}(s, x), \\ \mathcal{D}_{[\ell+1;\ell]}^{(3)}(s, x) &= s^{\ell+1} (\chi_\ell(x) - s\chi_{\ell-1}(x)) P^{(3)}(s, x), \\ \mathcal{D}_{[1;\frac{1}{2}]}^{(3)}(s, x) &= s(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(1 - sx)^{-1}(1 - sx^{-1})^{-1}, \\ \mathcal{D}_{[\frac{1}{2};0]}^{(3)}(s, x) &= s^{\frac{1}{2}}(1 + s)(1 - sx)^{-1}(1 - sx^{-1})^{-1}, \end{aligned} \quad (3.42)$$

exhaust all characters of the unitary irreducible representations of $SO(3, 2)$. Analogous formulae are to be found in [12].

For $SO(4) \simeq SU(2) \otimes SU(2)$ we have that

$$C_{(\ell_1, \ell_2)}^{(4)}(x_1, x_2) = C_j(x)C_{\bar{j}}(y), \quad \text{for } \ell_1 = j + \bar{j}, \quad \ell_2 = j - \bar{j}, \quad x_1 = x^{\frac{1}{2}}y^{\frac{1}{2}}, \quad x_2 = x^{\frac{1}{2}}y^{-\frac{1}{2}}, \quad (3.43)$$

i.e. the Verma module character with dominant highest weight (ℓ_1, ℓ_2) may be expressed as a product of two $SU(2)$ Verma module characters with highest weights j, \bar{j} . The characters of unitary irreducible representations of $SO(4, 2)$ are given by,

$$\begin{aligned} \mathcal{A}_{[\Delta;j,\bar{j}]}^{(4)}(s, x, y) &= s^\Delta \chi_j(x) \chi_{\bar{j}}(y) P^{(4)}(s, x, y), \\ \mathcal{D}_{[j+\bar{j}+2;j,\bar{j}]}^{(4)}(s, x, y) &= s^{j+\bar{j}+2} \left(\chi_j(x) \chi_{\bar{j}}(y) - s \chi_{j-\frac{1}{2}}(x) \chi_{\bar{j}-\frac{1}{2}}(y) \right) P^{(4)}(s, x, y), \\ \mathcal{D}_{[j+1;j]_+}^{(4)}(s, x, y) &= s^{j+1} \left(\chi_j(x) - s \chi_{j-\frac{1}{2}}(x) \chi_{\frac{1}{2}}(y) + s^2 \chi_{j-1}(x) \right) P^{(4)}(s, x, y), \\ \mathcal{D}_{[\bar{j}+1;\bar{j}]_-}^{(4)}(s, x, y) &= s^{\bar{j}+1} \left(\chi_{\bar{j}}(y) - s \chi_{\bar{j}-\frac{1}{2}}(y) \chi_{\frac{1}{2}}(x) + s^2 \chi_{\bar{j}-1}(y) \right) P^{(4)}(s, x, y). \end{aligned} \quad (3.44)$$

⁴ In terms of the notation employed in [6] these character formulae agree for $x \rightarrow \beta^2$, $s \rightarrow \alpha^2$. In the nomenclature of [6], the representations $\mathcal{D}_{[1;\frac{1}{2}]}^{(3)}$ and $\mathcal{D}_{[\frac{1}{2};0]}^{(3)}$ correspond to the ‘Di’ and ‘Rac’ singleton representations, respectively.

Here we have written $P^{(4)}(s, x_1, x_2) \rightarrow P^{(4)}(s, x, y)$ for x_1, x_2 as in (3.43). This reproduces the results for character formulae in four dimensions found in [4]. Similar formulae are to be found in [13].

Free fields have conformal dimension $\ell + \frac{1}{2}d - 1$ and belong to the $(\ell, \dots, \ell, \pm\ell)$ representation of $SO(2r)$ for any $\ell \in \frac{1}{2}\mathbb{N}$ in $d = 2r$ dimensions and the $(\ell, \dots, \ell, \ell)$ representation of $SO(2r+1)$ for $\ell = 0, \frac{1}{2}$ in $d = 2r+1$ dimensions [21]. The corresponding characters for even dimensions are $\mathcal{D}_{[\ell+r-1; \ell] \pm}^{(2r)}(s, x)$ in (3.26) along with $\mathcal{D}_{[r-1; 0]}^{(2r)}(s, x)$ in (3.27) for the scalar case. For odd dimensions the corresponding characters are $\mathcal{D}_{[r; \frac{1}{2}]}^{(2r+1)}(s, x)$ of (3.34) and $\mathcal{D}_{[r-\frac{1}{2}; 0]}^{(2r+1)}(s, x)$ in (3.33).

The characters (3.25), (3.32) for the special case of $p = 1$ and $\ell_1 \equiv \ell, \ell_2 = \dots = \ell_r = 0$ correspond to conserved symmetric traceless tensor-field representations of the conformal group, $T_{\mu_1 \dots \mu_\ell} = T_{(\mu_1 \dots \mu_\ell)}$, $T^\mu_{\mu\mu_2 \dots \mu_\ell} = \partial^{\mu_1} T_{\mu_1 \dots \mu_\ell} = 0$. These have conformal dimension $d + \ell - 2$ in d dimensions and examples are the conserved vector current for $\ell = 1$ and energy momentum tensor for $\ell = 2$.

4. Product formulae

We now turn to the determination of the decomposition of products of unitary irreducible representations of the conformal group into other unitary irreducible representations.

Product formulae: four dimensions

We illustrate for the $SO(4, 2)$ case first. For these purposes we first note a useful identity, namely,

$$P^{(4)}(s, x, y) = \sum_{p, q=0}^{\infty} s^{2p+q} \chi_{\frac{1}{2}q}(x) \chi_{\frac{1}{2}q}(y). \quad (4.1)$$

With (4.1) we may now easily determine the products of unitary irreducible representations of the conformal group. Using the usual decomposition of products of $SU(2)$ characters,

$$\chi_{j \otimes j'}(x) \equiv \chi_j(x) \chi_{j'}(x) = \sum_{q=|j-j'|}^{j+j'} \chi_q(x), \quad (4.2)$$

and (4.1), we notice that

$$\mathcal{D}_{[j+1; j]+}^{(4)}(s, x, y) = \sum_{q=0}^{\infty} s^{q+j+1} \chi_{j+\frac{1}{2}q}(x) \chi_{\frac{1}{2}q}(y). \quad (4.3)$$

Thus we may straightforwardly determine that,

$$\begin{aligned}
& \mathcal{D}_{[j+1,j]+}^{(4)}(s, x, y) \mathcal{D}_{[j'+1,j']_+}^{(4)}(s, x, y) \\
&= s^{j+j'+2} P(s, x, y) \left(\chi_{j \otimes j'}(x) + \sum_{q=1}^{\infty} s^q \left(\chi_{j+j'+\frac{1}{2}q}(x) \chi_{\frac{1}{2}q}(y) - s \chi_{j+j'+\frac{1}{2}q-\frac{1}{2}}(x) \chi_{\frac{1}{2}q-\frac{1}{2}}(y) \right) \right) \\
&= \mathcal{A}_{[j+j'+2;j \otimes j', 0]}^{(4)}(s, x, y) + \sum_{q=1}^{\infty} \mathcal{D}_{[j+j'+q+2;j+j'+\frac{1}{2}q, \frac{1}{2}q]}^{(4)}(s, x, y).
\end{aligned} \tag{4.4}$$

Similarly, using (4.2), (4.3), we may easily determine that,

$$\begin{aligned}
& \mathcal{D}_{[j+1,j]+}^{(4)}(s, x, y) \mathcal{D}_{[\bar{j}+1;\bar{j}]_+}^{(4)}(s, x, y) \\
&= s^{j+\bar{j}+2} P(s, x, y) \sum_{q=0}^{\infty} s^q \left(\chi_{j+\frac{1}{2}q}(x) \chi_{\bar{j}+\frac{1}{2}q}(y) - s \chi_{j+\frac{1}{2}q-\frac{1}{2}}(x) \chi_{\bar{j}+\frac{1}{2}q-\frac{1}{2}}(y) \right) \\
&= \sum_{q=0}^{\infty} \mathcal{D}_{[j+\bar{j}+q+2;j+\frac{1}{2}q, \bar{j}+\frac{1}{2}q]}^{(4)}(s, x, y).
\end{aligned} \tag{4.5}$$

Using (4.2), (4.3), we may also find that

$$\begin{aligned}
& \mathcal{D}_{[j+\bar{j}+2;j,\bar{j}]}^{(4)}(s, x, y) \mathcal{D}_{[j'+1,j']_+}^{(4)}(s, x, y) \\
&= \sum_{q=0}^{\infty} \left(\mathcal{A}_{[\Delta_q, j+j'+\frac{1}{2}q, \bar{j}+\frac{1}{2}q]}^{(4)}(s, x, y) + \mathcal{A}_{[\Delta_q, (j-\frac{1}{2}) \otimes (j'+\frac{1}{2}q-\frac{1}{2}), \bar{j}+\frac{1}{2}q]}^{(4)}(s, x, y) \right. \\
&\quad \left. + \mathcal{A}_{[\Delta_q, j+j'+\frac{1}{2}q, (\bar{j}-\frac{1}{2}) \otimes (\frac{1}{2}q-\frac{1}{2})]}^{(4)}(s, x, y) \right), \quad \Delta_q = j + j' + \bar{j} + q + 3,
\end{aligned} \tag{4.6}$$

and

$$\mathcal{A}_{[\Delta;j,\bar{j}]}^{(4)}(s, x, y) \mathcal{D}_{[j'+1,j']_+}^{(4)}(s, x, y) = \sum_{q=0}^{\infty} \mathcal{A}_{[\Delta+j'+q+1;j \otimes (j'+\frac{1}{2}q), \bar{j} \otimes \frac{1}{2}q]}^{(4)}(s, x, y), \tag{4.7}$$

which exhausts all products involving $\mathcal{D}_{[j+1;j]+}^{(4)}(s, x, y)$. Those involving $\mathcal{D}_{[\bar{j}+1;\bar{j}]_+}^{(4)}(s, x, y)$ may be obtained by the exchange $x \leftrightarrow y$ above noting

$$\begin{aligned}
\mathcal{D}_{[\bar{j}+1;\bar{j}]_+}^{(4)}(s, x, y) &= \mathcal{D}_{[\bar{j}+1;\bar{j}]+}^{(4)}(s, y, x), \quad \mathcal{D}_{[j+\bar{j}+2;j,\bar{j}]}^{(4)}(s, x, y) = \mathcal{D}_{[j+\bar{j}+2;j,\bar{j}]}^{(4)}(s, y, x), \\
\mathcal{A}_{[\Delta;j,\bar{j}]}^{(4)}(s, x, y) &= \mathcal{A}_{[\Delta;\bar{j},j]}^{(4)}(s, y, x).
\end{aligned} \tag{4.8}$$

Similarly, using (4.1) directly, we have that

$$\mathcal{A}_{[\Delta;j,\bar{j}]}^{(4)}(s, y, x) \mathcal{A}_{[\Delta';j',\bar{j}']}^{(4)}(s, y, x) = \sum_{p,q=0}^{\infty} \mathcal{A}_{[\Delta+\Delta'+2p+q;j \otimes j' \otimes \frac{1}{2}q, \bar{j} \otimes \bar{j}' \otimes \frac{1}{2}q]}^{(4)}(s, x, y). \tag{4.9}$$

We may note that each of these product formulae is compatible with the ‘blind’ partition functions,

$$\begin{aligned}
\mathcal{A}_{[\Delta;j,\bar{j}]}^{(4)}(s, 1, 1) &= \frac{s^\Delta}{(s-1)^4} (2j+1)(2\bar{j}+1), \\
\mathcal{D}_{[j+\bar{j}+2;j,\bar{j}]}^{(4)}(s, 1, 1) &= \frac{s^{j+\bar{j}+2}}{(s-1)^4} ((2j+1)(2\bar{j}+1) - 4sj\bar{j}), \\
\mathcal{D}_{[j+1;j]_+}^{(4)}(s, 1, 1) &= \mathcal{D}_{[j+1;j]_-}^{(4)}(s, 1, 1) = \frac{s^{j+1}}{(s-1)^3} (-(2j+1) + s(2j-1)).
\end{aligned} \tag{4.10}$$

As we see, the $d = 4$ cases are relatively simple when we use expansion formulae of the type (4.1), (4.3) to expand one of the characters in the product of two. The general cases which we consider now are also made simpler with analogous expansion formulae.

Product formulae: even dimensions

Useful for finding product formulae for the $SO(2r, 2)$ conformal group is the following expansion of $P(s, x)$ in terms of $SO(2r)$ characters, namely,

$$P^{(2r)}(s, x) = \sum_{p,q=0}^{\infty} s^{2p+q} \chi_{(q,0,\dots,0)}^{(2r)}(x), \tag{4.11}$$

where

$$\chi_{(q,0,\dots,0)}^{(2r)}(x) = \frac{1}{2} \det[x_i^{k_j} + x_i^{-k_j}] \Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1}, \tag{4.12}$$

with $k_1 = q + r - 1$, $k_j = r - j$, $j > 1$, for $\Delta(x)$ being the Vandermonde determinant,

$$\Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i). \tag{4.13}$$

The latter expression for the character comes from appendix B where also expressions for more general characters of $SO(d)$ in even and odd dimensions are given.

Analogously to (4.3) we have for (3.26) that

$$\mathcal{D}_{[\ell+r-1;\ell]_{\pm}}^{(2r)}(s, x) = \sum_{q=0}^{\infty} s^{\ell+r+q-1} \chi_{(\ell+q,\ell,\dots,\ell,\pm\ell)}^{(2r)}(x), \tag{4.14}$$

which we prove in appendix D.

More generally for the $p = r - j$ case of (3.25) (note that (4.14) encapsulates the $p = r$ case) we have, for $\ell > \ell_1 > \dots > |\ell_j|$,

$$\begin{aligned}
&\mathcal{D}_{[\ell+r+j-1;\ell,\ell_1,\dots,\ell_j]}^{(2r)}(s, x) \\
&= \sum_{p_1,\dots,p_j,q=0}^{\infty} \sum_{i_1=-\frac{1}{2}p_1}^{\frac{1}{2}p_1} \dots \sum_{i_j=-\frac{1}{2}p_j}^{\frac{1}{2}p_j} s^{\ell+r+j+q+p_1+\dots+p_j-1} \chi_{(\ell+q,\ell,\dots,\ell,\ell_1+2i_1,\dots,\ell_j+2i_j)}^{(2r)}(x),
\end{aligned} \tag{4.15}$$

which we also prove in appendix D. Note that the weights $(\ell+q, \ell, \dots, \ell, \ell_1+2i_1, \dots, \ell_j+2i_j)$ may lie outside the dominant Weyl chamber, i.e. not satisfy (3.1), for particular ℓ, ℓ_i . However for such weights we may use that $\chi_{\underline{\ell}'}^{(2r)}(x) = \text{sgn}(w) \chi_{\underline{\ell}'w}^{(2r)}(x)$, for some $w \in \mathcal{W}_{2r}$, to relate such characters to the character with dominant integral highest weight $\underline{\ell}'w$.

A notable simplification to (4.15) occurs for the $p = 1$ case of (3.25) for $\ell_1 \equiv \ell, \ell_2 = \dots = \ell_r = 0$ which corresponds to conserved symmetric traceless tensor representations of the conformal group. In this case we obtain that, for $d > 4$,

$$\mathcal{D}_{[\ell+2r-2; \ell, 0, \dots, 0]}^{(2r)}(s, x) = \sum_{p, q=0}^{\infty} \sum_{k=0}^{\ell} s^{\ell+2r+2p+q-2} \chi_{(q+k, \ell-k, 0, \dots, 0)}^{(2r)}(x). \quad (4.16)$$

We now discuss products involving the representations in (4.14) which contain the ‘truncated’ representations in (3.25) for $p = 1$ namely, $\mathcal{D}_{[\ell_1+2r-2, \ell_1, \dots, \ell_r]}^{(2r)}$.

For $d = 6$, for example, we may find using (4.14), (3.25) for $p = 1$ and (3.26) that

$$\begin{aligned} & \mathcal{D}_{[\ell+2; \ell]_+}^{(6)}(s, x) \mathcal{D}_{[\ell'+2; \ell']_-}^{(6)}(s, x) \\ &= \mathcal{A}_{[\ell+\ell'+4; (\ell, \ell, \ell) \otimes (\ell', \ell', -\ell')]}^{(6)}(s, x) + \sum_{q=1}^{\infty} \mathcal{D}_{[\ell+\ell'+q+4; \ell+\ell'+q, \ell+\ell', \ell-\ell']}^{(6)}(s, x), \end{aligned} \quad (4.17)$$

and that

$$\mathcal{D}_{[\ell+2; \ell]_{\pm}}^{(6)}(s, x) \mathcal{D}_{[\ell'+2; \ell']_{\pm}}^{(6)}(s, x) = \sum_{q=0}^{\infty} \sum_{t=|\ell-\ell'|}^{\ell+\ell'} \mathcal{D}_{[\ell+\ell'+q+4; \ell+\ell'+q, t, \pm t]}^{(6)}(s, x). \quad (4.18)$$

Here and in the following we are using the short-hand notation, for $r = [\frac{1}{2}d]$,

$$\chi_{(\ell_1, \dots, \ell_r) \otimes (\ell'_1, \dots, \ell'_r)}^{(d)}(x) \equiv \chi_{(\ell_1, \dots, \ell_r)}^{(d)}(x) \chi_{(\ell'_1, \dots, \ell'_r)}^{(d)}(x), \quad (4.19)$$

in $\mathcal{A}_{[\Delta; \underline{\ell}]}(s, x)$. Of course (4.19) may be decomposed in terms of $SO(d)$ characters once we know how $\ell \otimes \ell'$ decomposes into irreducible representations.

More generally there is a distinction in such product formulae between the cases where the dimension is divisible by four or not so.

Explicitly, we have for $d = 4m$ that

$$\begin{aligned} & \mathcal{D}_{[\ell+2m-1; \ell]_+}^{(4m)}(s, x) \mathcal{D}_{[\ell'+2m-1; \ell']_-}^{(4m)}(s, x) \\ &= \sum_{q=0}^{\infty} \sum_{\substack{t_i \geq \ell-\ell' \\ t_i \geq t_{i+1}}}^{\ell+\ell'} \mathcal{D}_{[\ell+\ell'+q+4m-2; \ell+\ell'+q, t_1, t_1, t_2, t_2, \dots, t_{m-1}, t_{m-1}, \ell-\ell']}^{(4m)}(s, x), \end{aligned} \quad (4.20)$$

and

$$\begin{aligned}
& \mathcal{D}_{[\ell+2m-1;\ell]\pm}^{(4m)}(s, \mathbf{x}) \mathcal{D}_{[\ell'+2m-1;\ell']\pm}^{(4m)}(s, \mathbf{x}) \\
&= \mathcal{A}_{[\ell+\ell'+4m-2;(\ell,\dots,\pm\ell)\otimes(\ell',\dots,\pm\ell')]}^{(4m)}(s, \mathbf{x}) \\
&+ \sum_{q=1}^{\infty} \sum_{\substack{t_i \geq |\ell-\ell'| \\ t_i \geq t_{i+1}}}^{\ell+\ell'} \mathcal{D}_{[\ell+\ell'+q+4m-2;\ell+\ell'+q,\ell+\ell',t_1,t_1,t_2,t_2,\dots,t_{m-1},\pm t_{m-1}]}^{(4m)}(s, \mathbf{x}),
\end{aligned} \tag{4.21}$$

while for $d = 4m + 2$ we have that

$$\begin{aligned}
& \mathcal{D}_{[\ell+2m;\ell]+}^{(4m+2)}(s, \mathbf{x}) \mathcal{D}_{[\ell'+2m;\ell']-}^{(4m+2)}(s, \mathbf{x}) \\
&= \mathcal{A}_{[\ell+\ell'+4m;(\ell,\dots,\ell)\otimes(\ell',\dots,-\ell')]}^{(4m+2)}(s, \mathbf{x}) \\
&+ \sum_{q=1}^{\infty} \sum_{\substack{t_i \geq \ell-\ell' \\ t_i \geq t_{i+1}}}^{\ell+\ell'} \mathcal{D}_{[\ell+\ell'+q+4m;\ell+\ell'+q,\ell+\ell',t_1,t_1,t_2,t_2,\dots,t_{m-1},t_{m-1},\ell-\ell']}^{(4m+2)}(s, \mathbf{x}),
\end{aligned} \tag{4.22}$$

and

$$\mathcal{D}_{[\ell+2m;\ell]\pm}^{(4m+2)}(s, \mathbf{x}) \mathcal{D}_{[\ell'+2m;\ell']\pm}^{(4m+2)}(s, \mathbf{x}) = \sum_{q=0}^{\infty} \sum_{\substack{t_i \geq |\ell-\ell'| \\ t_i \geq t_{i+1}}}^{\ell+\ell'} \mathcal{D}_{[\ell+\ell'+q+4m;\ell+\ell'+q,t_1,t_1,t_2,t_2,\dots,t_m,\pm t_m]}^{(4m+2)}(s, \mathbf{x}). \tag{4.23}$$

A special case of the previous is the product involving the character $\mathcal{D}_{[r-1;0]}^{(2r)}$ in (3.27), corresponding to a free scalar field, for which we have

$$\mathcal{D}_{[r-1;0]}^{(2r)}(s, \mathbf{x}) \mathcal{D}_{[\ell+r-1;\ell]\pm}^{(2r)}(s, \mathbf{x}) = \sum_{q=0}^{\infty} \mathcal{D}_{[\ell+q+2r-2;\ell+q,\ell,\dots,\pm\ell]}^{(2r)}(s, \mathbf{x}). \tag{4.24}$$

Another special case of the above contains a result first found by Vasiliev [10] which generalises a well known result by Flato and Fronsdal [6] in three dimensions to even dimensions $d = 2r$. This result involves products of the representation corresponding to the free Dirac spinor,

$$\begin{aligned}
Di^{(2r)}(s, \mathbf{x}) &\equiv \mathcal{D}_{[r-\frac{1}{2},\frac{1}{2}]_+}^{(2r)}(s, \mathbf{x}) + \mathcal{D}_{[r-\frac{1}{2},\frac{1}{2}]_-}^{(2r)}(s, \mathbf{x}) \\
&= s^{r-\frac{1}{2}}(1-s)(\chi_{(\frac{1}{2},\dots,\frac{1}{2},\frac{1}{2})}^{(2r)}(\mathbf{x}) + \chi_{(\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2})}^{(2r)}(\mathbf{x}))P(s, \mathbf{x}) \\
&= s^{r-\frac{1}{2}}(1-s)(x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}}) \dots (x_r^{\frac{1}{2}} + x_r^{-\frac{1}{2}})P(s, \mathbf{x}).
\end{aligned} \tag{4.25}$$

Using the above product formulae we may show that

$$\begin{aligned}
& Di^{(2r)}(s, x) Di^{(2r)}(s, x) \\
&= 2\mathcal{A}_{[2r-1, 0, \dots, 0]}^{(2r)} \\
&+ 2 \sum_{q=0}^{\infty} \left(\mathcal{D}_{[2r+q-1, q+1, 1, 1, \dots, 1, 0]}^{(2r)}(s, x) + \mathcal{D}_{[2r+q-1, q+1, 1, 1, \dots, 1, 0, 0]}^{(2r)}(s, x) \right. \\
&\quad \left. + \dots + \mathcal{D}_{[2r+q-1, q+1, 1, 0, \dots, 0, 0]}^{(2r)}(s, x) + \mathcal{D}_{[2r+q-1, q+1, 0, 0, \dots, 0, 0]}^{(2r)}(s, x) \right) \\
&+ \sum_{q=0}^{\infty} \left(\mathcal{D}_{[2r+q-1, q+1, 1, 1, \dots, 1, 1]}^{(2r)}(s, x) + \mathcal{D}_{[2r+q-1, q+1, 1, 1, \dots, 1, -1]}^{(2r)}(s, x) \right),
\end{aligned} \tag{4.26}$$

regardless of whether $d = 2r$ is divisible by four or not so. The latter matches Vasiliev's result.

Product formulae: odd dimensions

For $SO(2r+1, 2)$ we have that

$$P^{(2r+1)}(s, x) = \sum_{p, q=0}^{\infty} s^{2p+q} \chi_{(q, 0, \dots, 0)}^{(2r+1)}(x), \tag{4.27}$$

where

$$\begin{aligned}
\chi_{(q, 0, \dots, 0)}^{(2r+1)}(x) &= \frac{1}{2} \det[x_i^{k_j} - x_i^{-k_j}] \Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1} \\
&\times (x_1^{\frac{1}{2}} - x_1^{-\frac{1}{2}})^{-1} \dots (x_r^{\frac{1}{2}} - x_r^{-\frac{1}{2}})^{-1},
\end{aligned} \tag{4.28}$$

with $k_1 = q + \frac{1}{2} + r - 1$, $k_j = \frac{1}{2} + r - j$, $j > 1$.

From the results of appendix D, we have the following expansions for the free spinor case of (3.34) and the free scalar case of (3.33), namely,

$$\mathcal{D}_{[r; \frac{1}{2}]}^{(2r+1)}(s, x) = \sum_{q=0}^{\infty} s^{r+q} \chi_{(\frac{1}{2}+q, \frac{1}{2}, \dots, \frac{1}{2})}^{(2r+1)}(x), \tag{4.29}$$

for the free spinor case and

$$\mathcal{D}_{[r-\frac{1}{2}; 0]}^{(2r+1)}(s, x) = \sum_{q=0}^{\infty} s^{r+q-\frac{1}{2}} \chi_{(q, 0, \dots, 0)}^{(2r+1)}(x), \tag{4.30}$$

for the free scalar case.

For (3.32) and $p = r - j$ we have that, for $\ell > \ell_1 > \dots > \ell_j$,

$$\begin{aligned}
& \mathcal{D}_{[\ell+r+j; \ell, \ell_1, \dots, \ell_j]}^{(2r+1)}(s, x) \\
&= \sum_{p_1, \dots, p_j, q, t=0}^{\infty} \sum_{i_1=-\frac{1}{2}p_1}^{\frac{1}{2}p_1} \dots \sum_{i_j=-\frac{1}{2}p_j}^{\frac{1}{2}p_j} s^{\ell+r+j+q+t+p_1+\dots+p_j} \chi_{(\ell+q, \ell, \dots, \ell, \ell_1+2i_1, \dots, \ell_j+2i_j)}^{(2r+1)}(x),
\end{aligned} \tag{4.31}$$

which we show in appendix D. Again, the weights $(\ell + q, \ell, \dots, \ell, \ell_1 + 2i_1, \dots, \ell_j + 2i_j)$ may lie outside the dominant Weyl chamber, i.e. not satisfy (3.2), for particular ℓ, ℓ_i . For such weights we may use that $\chi_{\underline{\ell}'}^{(2r+1)}(x) = \text{sgn}(w) \chi_{\underline{\ell}'w}^{(2r+1)}(x)$, for some $w \in \mathcal{W}_{2r+1}$, to relate such characters to those with dominant integral highest weights $\underline{\ell}'w$.

Just as for even dimensions in (4.16) a simplification to (4.31) occurs for the $p = 1$ case of (3.32) for $\ell_1 \equiv \ell, \ell_2 = \dots = \ell_r = 0$ for which,

$$\mathcal{D}_{[\ell+2r-1; \ell, 0, \dots, 0]}^{(2r+1)}(s, x) = \sum_{p, q=0}^{\infty} \sum_{k=0}^{\ell} s^{\ell+2r+2p+q-1} \chi_{(q+k, \ell-k, 0, \dots, 0)}^{(2r+1)}(x). \quad (4.32)$$

Regarding products of free representations, we may determine that, using (3.32) for $p = 1$,

$$\begin{aligned} \mathcal{D}_{[r; \frac{1}{2}]}^{(2r+1)}(s, x) \mathcal{D}_{[r; \frac{1}{2}]}^{(2r+1)}(s, x) &= \mathcal{A}_{[2r; 0, \dots, 0]}(s, x) \\ &+ \sum_{q=0}^{\infty} \left(\mathcal{D}_{[2r+q; q+1, 1, \dots, 1]}^{(2r+1)}(s, x) + \mathcal{D}_{[2r+q; q+1, 1, \dots, 1, 0]}^{(2r+1)}(s, x) + \dots + \mathcal{D}_{[2r+q; q+1, 0, \dots, 0]}^{(2r+1)}(s, x) \right), \end{aligned} \quad (4.33)$$

and

$$\mathcal{D}_{[r; \frac{1}{2}]}^{(2r+1)}(s, x) \mathcal{D}_{[r-\frac{1}{2}; 0]}^{(2r+1)}(s, x) = \sum_{q=0}^{\infty} \mathcal{D}_{[2r+q-\frac{1}{2}; q+\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}]}^{(2r+1)}(s, x), \quad (4.34)$$

along with

$$\mathcal{D}_{[r-\frac{1}{2}; 0]}^{(2r+1)}(s, x) \mathcal{D}_{[r-\frac{1}{2}; 0]}^{(2r+1)}(s, x) = \sum_{q=0}^{\infty} \mathcal{D}_{[2r+q-1; q, 0, \dots, 0]}^{(2r+1)}(s, x), \quad (4.35)$$

which generalise similar formulae obtained in [6] to odd dimensions.

5. Partition functions

As a partial check of the character formulae corresponding to free fields, we will compare them to partition functions which have been obtained by various authors in conformally invariant theories on $S^1 \times S^{d-1}$. For these cases, the single particle partition function for a local free operator F may expressed by

$$Y_F^{(d)}(s) = \sum_{q=0}^{\infty} n_{F,q}^{(d)} s^{\Delta_0 + q}, \quad (5.1)$$

where $n_{F,q}^{(d)}$ enumerates the descendants of F in the flat background \mathbb{R}^d . For even d , the form (4.14) for the character formula corresponding to such free fields allows us to obtain $Y_F^{(d)}(s)$ directly when we set $x_1, \dots, x_{\frac{1}{2}d} = 1$. Thus we easily find that

$$Y_{F_{\pm}}^{(d)}(s) = \sum_{q=0}^{\infty} n_{F_{\pm},q}^{(d)} s^{\ell + \frac{1}{2}d + q - 1}, \quad (5.2)$$

where for $\ell \neq 0$ (from appendix B)

$$\begin{aligned} n_{F_{\pm},q}^{(d)} &= \chi_{(\ell+q,\ell,\dots,\ell,\pm\ell)}^{(d)}(1, \dots, 1) = \dim(\mathcal{I}_{(\ell+q,\ell,\dots,\ell)}^{(d)}) \\ &= 2^{\frac{1}{2}d-1} \prod_{i=1}^{\frac{1}{2}d-1} \frac{1}{(d-2i)!} (q+i)(2\ell+q+d-2-i) \prod_{2 \leq k < j \leq \frac{1}{2}d} (j-k)(2\ell+d-j-k), \end{aligned} \quad (5.3)$$

while for the scalar field

$$\begin{aligned} n_{S,q}^{(d)} &= \chi_{(q,0,\dots,0)}^{(d)}(1, \dots, 1) \\ &= \frac{2q+d-2}{q+d-2} \binom{q+d-2}{q}, \end{aligned} \quad (5.4)$$

which is the dimension of the rank- q symmetric traceless tensor representation of $SO(d)$. (This agrees with a similar formula in [14].) For chiral Weyl fermions we find, from (5.3) for $\ell = \frac{1}{2}$,

$$n_{f_{\pm},q}^{(d)} = 2^{\frac{1}{2}d} \frac{1}{q!} (q+1)(q+2) \dots (q+d-2). \quad (5.5)$$

Similarly for the $\frac{1}{2}d$ -form field strength, from (5.3) for $\ell = 1$,

$$n_{V_{\pm},q}^{(d)} = \frac{d}{2(2q+d)} \binom{d}{\frac{1}{2}d} \binom{q+d-1}{q}. \quad (5.6)$$

Note that it may be easily checked that these occupancy numbers agree with those obtained in [15]. for $d = 4, 6$ (where for $d = 4$ then $Y_{V^+}^{(4)}(s) + Y_{V^-}^{(4)}(s)$ is the single particle partition function of the Maxwell field).

For bosonic F the multi-particle partition function is given by

$$Z_F^{(d)}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Y_F^{(d)}(s^n)\right) = \prod_{q=0}^{\infty} (1 - s^{\Delta_0+q})^{-n_{F,q}^{(d)}}, \quad (5.7)$$

while for fermionic F it is given by

$$Z_F^{(d)}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} Y_F^{(d)}(s^n)\right) = \prod_{q=0}^{\infty} (1 + s^{\Delta_0+q})^{n_{F,q}^{(d)}}. \quad (5.8)$$

It is easy to check that (5.7), (5.8) for the scalar, Weyl fermion and field strength cases above match the results of [15] for $d = 4, 6$.

Performing the summation in (5.2) for the scalar, Weyl fermion and $\frac{1}{2}d$ -form field-strength cases we find

$$\begin{aligned} Y_S^{(d)}(s) &= (s+1) \frac{s^{\frac{1}{2}d-1}}{(1-s)^{d-1}}, \\ Y_{f\pm}^{(d)}(s) &= 2^{\frac{1}{2}d} \frac{s^{\frac{1}{2}(d-1)}}{(1-s)^{d-1}}, \\ Y_{V\pm}^{(d)}(s) &= \frac{d!}{2(\frac{1}{2}d)!^2} \frac{s^{\frac{1}{2}d}}{(1-s)^{d-1}} F(1, -\frac{1}{2}d+1; \frac{1}{2}d+1; s), \end{aligned} \quad (5.9)$$

(where $F(a, b; c; x)$ is the usual hypergeometric function) which may be read off directly from (3.26), (3.27) for $x_i = 1$. This form may be directly compared to similar results in [16] whereby the formulae agree for the scalar and Weyl fermion cases.

It is not difficult to compute the first couple of numbers (5.6) for the self-dual $r = \frac{1}{2}d$ form field strength which we denote by $F_{\mu_1 \dots \mu_r} = {}^*F_{\mu_1 \dots \mu_r}$. For $q = 0$ this number just counts the number of independent components in $F_{\mu_1 \dots \mu_r}$. Anti-symmetry in the indices implies $\binom{2r}{r}$ independent components which is reduced by a factor of a half due to self-duality. For $q = 1$ (5.6) counts the number of first-order descendants, $\partial_\mu F_{\mu_1 \dots \mu_r}$. This will be $r \binom{2r}{r}$ less the number of constraint equations $\partial^{\mu_1} F_{\mu_1 \dots \mu_r} = 0$ which is $\binom{2r}{r-1}$.

As a further example and check of our formulae, we consider the single particle partition function for rank- ℓ symmetric traceless tensor fields $T_{\mu_1 \dots \mu_\ell}$ satisfying the constraint equation $\partial^{\mu_1} T_{\mu_1 \dots \mu_\ell} = 0$. The appropriate character formula in this case is given by (4.16). The corresponding occupation numbers, for $n_{F,q}^{(d)} \rightarrow N_{q,\ell}$ in (5.2), are given by

$$\begin{aligned} N_{q,\ell} &= \left(q(d-2)(2\ell+d-3) + (d-1)(\ell+d-3)(2\ell+d-2) \right) \\ &\quad \times \frac{1}{(d-1)(d-2)(d-3)} \binom{\ell+d-4}{\ell} \binom{q+d-2}{q}. \end{aligned} \quad (5.10)$$

To see this we may use (4.16) to write

$$N_{q,\ell} = \sum_{i=0}^{\lfloor \frac{1}{2}q \rfloor} t_{q-2i,\ell} \quad \text{for} \quad t_{q,\ell} = \sum_{k=0}^{\ell} \dim(\mathcal{I}_{(q+k,\ell-k,0,\dots,0)}^{(d)}). \quad (5.11)$$

Using the dimension formula in appendix B we may find that

$$\begin{aligned} t_{q,\ell} &= (\ell+q+d-3) \left(4\ell q + (d-3)(2\ell+2q+d-2) \right) \\ &\quad \times \frac{1}{(d-2)(d-3)^2} \binom{\ell+d-4}{\ell} \binom{q+d-4}{q}, \end{aligned} \quad (5.12)$$

and thence obtain (5.10) from (5.11).

Again it is not difficult to check that the first couple of numbers agree with expectations. We may easily show that $N_{\ell,0} = \dim(\mathcal{I}_{(\ell,0,\dots,0)}^{(d)})$ given in (5.4) as expected. Also we may easily show that $N_{\ell,1} = d \dim(\mathcal{I}_{(\ell,0,\dots,0)}^{(d)}) - \dim(\mathcal{I}_{(\ell-1,0,\dots,0)}^{(d)})$ which is the number of first-order descendants $\partial_\mu T_{\mu_1 \dots \mu_\ell}$ reduced by the number of constraint equations coming from the conservation condition. More generally

$$N_{\ell,q} = \binom{q+d-1}{q} \dim(\mathcal{I}_{(\ell,0,\dots,0)}^{(d)}) - \binom{q+d-2}{q-1} \dim(\mathcal{I}_{(\ell-1,0,\dots,0)}^{(d)}), \quad (5.13)$$

which may be easily seen as the descendants at level q are given by $\partial_{\nu_1} \partial_{\nu_2} \dots \partial_{\nu_q} T_{\mu_1 \dots \mu_\ell}$ whose number of independent components is given by the first term in (5.13) which is reduced by the number of independent components in $\partial_{\nu_1} \partial_{\nu_2} \dots \partial_{\nu_{q-1}} \partial^{\mu_1} T_{\mu_1 \dots \mu_\ell}$ which vanishes by conservation.

More generally, we may use (4.15) in even dimensions and (4.29), (4.30), (4.31) in odd dimensions for $x_i = 1$ to determine the occupation numbers in the single particle partition function corresponding to fields whose conformal dimension saturates the unitarity bounds (3.8), (3.9). As mentioned before these are fields which satisfy certain conservation conditions which determines the particular unitarity bound.

Rotating quantum fields in an AdS_{d+1} background have been considered in [17]. Here the modes of a quantum field are supposed to have energies E and angular momenta j_i where $i = 1, \dots, [\frac{1}{2}d]$. For the boundary conformal field theory on $\mathbb{R} \times S^{d-1}$ the energies are related to the conformal dimension of conformal fields and their descendants via $E = \Delta$ assuming that the sphere has unit radius while j_i correspond to $SO(d)$ eigenvalues. Making the identification

$$s = e^{-\beta}, \quad x_i = e^{\beta \Omega_i}, \quad (5.14)$$

where $\beta = T^{-1}$ for T being the temperature and Ω_i denote chemical potentials for angular momenta, then it can be shown that the one particle boundary partition function $\sum_{E,j_i} e^{-\beta(E - \Omega_i j_i)}$ and character formula for the conformal field coincide. For instance, for a free scalar field, the character formula (3.27) obtained here agrees with the corresponding single particle partition function for the boundary conformal field theory obtained in [17] when we make the identification (5.14).

6. Acknowledgements

I am grateful to Nicolas Boulanger, Gary Gibbons, Paul Heslop and Hugh Osborn for discussions.

Appendix A. Character formulae for infinite dimensional irreducible modules of semi-simple Lie algebras

In this appendix we outline some results on character formulae of relevance for the main text. First we give basic definitions and notation.

The Weyl group \mathcal{W} is generated by mappings w_α , for α being a root, which on weight space give

$$w_\alpha(\Lambda) = \Lambda - (\alpha^\vee, \Lambda) \alpha, \quad \alpha^\vee = 2\alpha/(\alpha, \alpha). \quad (\text{A.1})$$

For $\Lambda = \beta$, another root, then $w_\alpha(\beta)$ is a reflection of β with respect to the hyperplane through the origin and perpendicular to α . Here (λ, μ) denotes the usual inner product on weight space between λ and μ . (In the Dynkin basis this is given by $(\lambda, \mu) = \sum_{i,j} \lambda_i G_{ij} \mu_j$ where λ_i, μ_i are Dynkin labels and $[G_{ij}]$ is the quadratic form matrix.) Any $w \in \mathcal{W}$ may be decomposed in terms of simple Weyl reflections $w_i \equiv w_{\alpha_i}$, for α_i being the simple roots, as $w = w_{i_1} \cdots w_{i_n}$ for some n which is generally not unique. However the signature of w defined, in the present case, by $\text{sgn}(w) = (-1)^n$ is uniquely defined. We denote by $\ell(w)$ the minimum number of w_i in the composition of w . Clearly $\text{sgn}(w) = (-1)^{\ell(w)}$.

The Weyl group divides the weight space into a family of open sets called Weyl chambers. These are simplicial cones defined by

$$\mathcal{H}_w = \{\lambda : (\alpha_i^\vee, w(\lambda)) > 0, \quad 1 \leq i \leq r\}, \quad (\text{A.2})$$

for $w \in \mathcal{W}$. The number of such equals the order of \mathcal{W} , $|\mathcal{W}|$. The weights lying on the boundary of the Weyl chambers are the points on the hyperplanes perpendicular to the roots $(\alpha_i^\vee, w(\lambda)) = 0$. In terms of Dynkin labels these are the weights having at least one vanishing Dynkin label.

The Weyl chamber corresponding to the identity of the Weyl group \mathcal{H}_1 is the fundamental or dominant Weyl chamber. In terms of Dynkin labels the weights in this chamber have strictly positive Dynkin labels. If all the Dynkin labels are non-negative and/or integers then we say the weight is dominant and/or integral.

Suppose we have some Lie algebra module V_Λ having highest weight Λ . The definition of the corresponding character we use is [5]

$$\text{Char}_\Lambda \equiv \sum_{\lambda \in V_\Lambda} \text{mult}_{V_\Lambda}(\lambda) e^\lambda, \quad (\text{A.3})$$

where $\text{mult}_{V_\Lambda}(\lambda)$, $\text{mult}_{V_\Lambda}(\Lambda) = 1$, denotes the multiplicity of the weight λ in the weight system of V_Λ . This is to be interpreted as a function on weight space satisfying

$$e^\lambda e^\mu = e^{\lambda+\mu}, \quad e^\lambda(\mu) = e^{(\lambda, \mu)}. \quad (\text{A.4})$$

Under the action of the Weyl group, for $w \in \mathcal{W}$,

$$w(e^\lambda) = e^{w(\lambda)}. \quad (\text{A.5})$$

For a unitary group and with $\text{mult}_{V_\Lambda}(\lambda)$ always finite we may recover a trace formula for the character by normalising each vector $v_\lambda \in V_\Lambda$ corresponding to the weight λ so that $\langle v_\lambda | v_\lambda \rangle = 1$. Then we may write,

$$\text{Char}_\Lambda(\mu) = \sum_{v_\lambda \in V_\Lambda} \langle v_\lambda | v_\lambda \rangle e^{(\lambda, \mu)} = \text{Tr}(e^{(H, \mu)}), \quad (\text{A.6})$$

where $(H, \mu) = \sum_{i,j} H_i G_{ij} \mu_j$ in the Dynkin basis, for example, for H_i being Cartan subalgebra elements with $H_i |v_\lambda\rangle = \lambda_i |v_\lambda\rangle$.

As an example, consider a Verma module \mathcal{V}_Λ with basis $\prod_{\alpha \in \Phi_-} E_\alpha^{n_\alpha} |\underline{\Lambda}\rangle^{\text{h.w.}}$ for Φ_- being negative roots and n_α being non-negative integers. For fixed n_α then the corresponding weight $\lambda_{(n_\alpha)} = \Lambda + \sum_{\alpha \in \Phi_-} n_\alpha \alpha$ has unit multiplicity in the weight system of \mathcal{V}_Λ . Thus we may write the character for the Verma module as

$$\mathcal{C}_\Lambda = \sum_{n_\alpha \geq 0} e^{\lambda_{(n_\alpha)}} = e^\Lambda \prod_{\alpha \in \Phi_-} (1 - e^\alpha)^{-1}. \quad (\text{A.7})$$

Note that a given weight λ has multiplicity given by $\mathcal{P}(\Lambda - \lambda)$ where $\mathcal{P}(\mu)$ counts the number of ways in which the weight μ may be written as a linear combination of positive roots with non-negative integer coefficients.

We may easily also show that

$$w(\mathcal{C}_\Lambda) = \text{sgn}(w) \mathcal{C}_{\Lambda^w}, \quad \Lambda^w = w(\Lambda + \rho) - \rho, \quad (\text{A.8})$$

for any $w \in \mathcal{W}$.

The character χ_Λ of an infinite dimensional irreducible module \mathcal{I}_Λ of a semi-simple Lie algebra has been written down long ago [11]. For the highest weight Λ not being dominant integral then \mathcal{I}_Λ is infinite dimensional. Otherwise \mathcal{I}_Λ is finite dimensional and the character is given by the well known Weyl character formula. Before we give the result of [11] we quote a number of results which give insight into the structure of infinite dimensional irreducible modules.

Concerning Verma modules (which in [11] are called ‘elementary representations’), the first result we recount is that if Λ^w , $w \in \mathcal{W}$ is not a weight of \mathcal{V}_Λ then \mathcal{V}_Λ itself is irreducible. This is the simplest case and an example is \mathcal{V}_ℓ for $SO(3)$ with ℓ being a

negative half integer. We now give the conditions for \mathcal{V}_Λ to contain sub-modules $\mathcal{V}_{\Lambda'}$ and so to be reducible.

We define a partial ordering on weights so that $\Lambda' \prec \Lambda$ if and only if $\Lambda - \Lambda' = \Pi$ for some $\Pi = \sum_{\alpha \in \Phi_+} p_\alpha \alpha$ for p_α being non-negative integers, not all zero. A necessary condition for a Verma module \mathcal{V}_Λ to contain a sub-module $\mathcal{V}_{\Lambda'}$ is that $\Lambda' = \Lambda^w \prec \Lambda$ for some $w \in \mathcal{W}$, $w \neq 1$. A crucial result is a theorem in [22] which proves that a necessary and sufficient condition for \mathcal{V}_{Λ^w} to be a sub-module of \mathcal{V}_Λ is that there exist a sequence of positive roots β_1, \dots, β_K such that

$$w = w_{\beta_1} w_{\beta_2} \dots w_{\beta_K}, \quad (w_{\beta_k}^\vee, w_{\beta_{k+1}} w_{\beta_{k+2}} \dots w_{\beta_K} (\Lambda + \rho)) \in \mathbb{N}, \quad k = 1, \dots, K, \quad (\text{A.9})$$

where we define $w_{\beta_{K+1}} \equiv 1$. For Λ being a dominant integral weight then condition (A.9) holds for all Λ^w , $w \in \mathcal{W}$. This justifies the claim made in this paper that for Λ being dominant integral then Λ^w is a highest weight in \mathcal{V}_Λ for every $w \in \mathcal{W}$.

A key result proved by Verma [23] and recounted in [11] is the following. A Verma module contains those and only those irreducible representations $\mathcal{I}_{\Lambda'}$ for which $\mathcal{V}_{\Lambda'}$ is a sub-module of \mathcal{V}_Λ . Furthermore it contains $\mathcal{I}_{\Lambda'}$ at most once.

This results in the following formula for characters,

$$\mathcal{C}_\Lambda = \chi_\Lambda + \sum_{\substack{w \in \mathcal{W} \\ \Lambda^w \prec \Lambda}} \chi_{\Lambda^w}, \quad (\text{A.10})$$

where the sum runs over all w for which (A.9) holds.

Using these results and formulae for multiplicities of weights determined for Verma modules in terms of the function $\mathcal{P}(\mu)$ above a formula has been given for χ_Λ in [11]. This may be rewritten in the equivalent form

$$\chi_\Lambda = \mathcal{C}_\Lambda + \sum_{\substack{w \in \mathcal{W} \\ \Lambda^w \prec \Lambda}} \gamma_{\Lambda^w} \mathcal{C}_{\Lambda^w}, \quad (\text{A.11})$$

where each w satisfies condition (A.9) and where the integers γ_{Λ^w} are determined by a recurrence relation as follows.

Consider a sub-module \mathcal{V}_{Λ^w} of \mathcal{V}_Λ so that there is no other sub-module $\mathcal{V}_{\Lambda^{w'}}$ of \mathcal{V}_Λ containing \mathcal{V}_{Λ^w} in turn as a sub-module. Then in this case $\gamma_{\Lambda^w} = -1$.

For a sub-module \mathcal{V}_{Λ^w} of \mathcal{V}_Λ which is in turn contained in the sub-modules $\mathcal{V}_{\Lambda^{w'}}$ of \mathcal{V}_Λ , then in this case

$$\gamma_{\Lambda^w} = - \sum_{w'} \gamma_{\Lambda^{w'}} - 1, \quad (\text{A.12})$$

which determines γ_{Λ^w} in (A.11) recursively.

A simple example illustrates this formula. Consider Sl_3 whereby, for a weight $\Lambda = [a, b]$ with Dynkin labels a, b , the simple Weyl reflections are given by

$$w_1([a, b]) = [-a, a + b], \quad w_2([a, b]) = [a + b, -b]. \quad (\text{A.13})$$

The \mathcal{S}_3 Weyl group elements consist of $\{1, w_1, w_2, w_1w_2, w_2w_1, w_\alpha\}$ where $w_\alpha = w_1w_2w_1 = w_2w_1w_2$ is the Weyl reflection corresponding to the root $\alpha = \alpha_1 + \alpha_2$. The shifted Weyl reflections are given by $\Lambda^1 = \Lambda$, $\Lambda^{w_1} = \Lambda - (a + 1)\alpha_1$, $\Lambda^{w_2} = \Lambda - (b + 1)\alpha_2$, $\Lambda^{w_2w_1} = \Lambda - (a + 1)\alpha_1 - (a + b + 2)\alpha_2$, $\Lambda^{w_1w_2} = \Lambda - (a + b + 2)\alpha_1 - (b + 1)\alpha_2$, $\Lambda^{w_\alpha} = \Lambda - (a + b + 2)\alpha$. For \swarrow denoting a vector in the $-\alpha_1$ direction of length $a + 1$ and \searrow denoting such in the $-\alpha_2$ direction of length $b + 1$ then we may represent the weight system of the Verma module \mathcal{V}_Λ diagrammatically as follows

$$\begin{array}{ccc}
 & \Lambda & \\
 \swarrow & & \searrow \\
 \Lambda^{w_1} & & \Lambda^{w_2} \\
 \searrow & & \swarrow \\
 & \Lambda^{w_1w_2} & \Lambda^{w_2w_1} \\
 \searrow & & \swarrow \\
 & \Lambda^{w_\alpha} &
 \end{array} \quad (\text{A.14})$$

assuming Λ is dominant integral and where we have omitted any other weights occurring. In computing $\chi_{\Lambda^{w_1}}$, for example, we have that $\mathcal{V}_{\Lambda^{w_1w_2}}$ and $\mathcal{V}_{\Lambda^{w_2w_1}}$ are both sub-Verma-modules of $\mathcal{V}_{\Lambda^{w_1}}$ with $\gamma_{\Lambda^{w_1w_2}} = \gamma_{\Lambda^{w_2w_1}} = -1$. Also $\mathcal{V}_{\Lambda^{w_\alpha}}$ is a sub-Verma-module of both the latter and so $\gamma_{\Lambda^{w_\alpha}} = -(\gamma_{\Lambda^{w_1w_2}} + \gamma_{\Lambda^{w_2w_1}}) - 1 = 1$. Thus $\chi_{\Lambda^{w_1}} = \mathcal{C}_{\Lambda^{w_1}} - \mathcal{C}_{\Lambda^{w_2w_1}} - \mathcal{C}_{\Lambda^{w_1w_2}} + \mathcal{C}_{\Lambda^{w_\alpha}}$. In fact for all $\Lambda' = \Lambda^{w'}$ it is easy to check that $\chi_{\Lambda'} = \mathcal{C}_{\Lambda'} + \sum_{\substack{w \in \mathcal{S}_3 \\ \Lambda' w \prec \Lambda'}} \text{sgn}(w) \mathcal{C}_{\Lambda' w}$.

Other properties of \mathcal{I}_Λ which are useful for what follows concern symmetry under Weyl group reflections. For all Dynkin labels Λ_i of the weight Λ which are non-negative integers then the sub-group \mathcal{W}' generated by the corresponding simple Weyl reflections w_i is the maximal symmetry group of the weight system of \mathcal{I}_Λ . Furthermore $\mathcal{V}_{\Lambda^{w'}}$ is a sub-module of \mathcal{V}_Λ for all $w' \in \mathcal{W}'$, $w' \neq 1$ (since all w' satisfy condition (A.9)). In terms of characters we have that $w'(\chi_\Lambda) = \chi_\Lambda$ for every $w' \in \mathcal{W}'$.

For Λ being dominant integral then this symmetry group is of course the Weyl group itself and $\mathcal{W}' = \mathcal{W}$. In this case we from (A.11) that

$$\chi_\Lambda = \sum_{w \in \mathcal{W}} \text{sgn}(w) \mathcal{C}_{\Lambda^w} = \prod_{\alpha \in \Phi_-} (1 - e^\alpha)^{-1} \sum_{w \in \mathcal{W}} \text{sgn}(w) e^{w(\Lambda + \rho) - \rho}, \quad (\text{A.15})$$

the usual Weyl character formula. This follows as symmetry under \mathcal{W} determines $\gamma_{\Lambda^w} = \text{sgn}(w)$ in this case. Using (A.8) we may find that the Weyl character may be rewritten as

$$\chi_{\Lambda} = \sum_{w \in \mathcal{W}} w(\mathcal{C}_{\Lambda}) \equiv \mathfrak{W}(\mathcal{C}_{\Lambda}). \quad (\text{A.16})$$

Due to the invariance of \mathcal{I}_{Λ} under the action of any $w \in \mathcal{W}$ then the Weyl symmetry operator \mathfrak{W} defined by (A.16) is obviously linear and idempotent on the vector space spanned by the characters of the Verma modules, $\mathfrak{W}^2 = |\mathcal{W}|\mathfrak{W}$.

Denoting by I the subset of labels for which Λ_i , $i \in I$, are non-negative integers, supposing we find that every sub-Verma-module highest weight $\Lambda^{w''}$, $w'' \in \mathcal{W}$ may be written in the form $\Lambda^{w''} = \Lambda^{w'w}$, $w \in \mathcal{W}$, $w' \in \mathcal{W}'$ for Λ^{w_i} , $i \in I$ being non-negative integers or $\Lambda^{w_i} = -1$ for some $i \in I$. We claim that the weights $\Lambda^{w''} \prec \Lambda$ are given by the disjoint union of the weights $\Lambda^{w'w}$ for every $w' \in \mathcal{W}'$. For the cases of $\Lambda^{w_i} = -1$ then the $\mathcal{C}_{\Lambda^{w'w}}$, $w' \in \mathcal{W}'$ cancel among themselves in χ_{Λ} .

Under this assumption (which holds in the cases considered in this paper) then using (A.11) we may write for the character

$$\chi_{\Lambda} = \mathcal{C}_{\Lambda} + \sum_{\substack{w' \in \mathcal{W}' \\ w' \neq 1}} \gamma_{\Lambda^{w'}} \mathcal{C}_{\Lambda^{w'}} + \sum_{\substack{w' \in \mathcal{W}' \\ w \in \mathcal{W}, w \neq 1}} \gamma_{\Lambda^{w'w}} \mathcal{C}_{\Lambda^{w'w}}, \quad (\text{A.17})$$

where the sum runs over only those $w \in \mathcal{W}$ which satisfy (A.9) and for which Λ^{w_i} , $i \in I$, is a non-negative integer or -1 . Using (A.8) and $w'(\chi_{\Lambda}) = \chi_{\Lambda}$, $w' \in \mathcal{W}'$ and the claims proved below we may show that $\gamma_{\Lambda^{w'}} = \text{sgn}(w')$, $\gamma_{\Lambda^{w'w}} = \gamma_{\Lambda^w} \text{sgn}(w')$ and that the $\mathcal{C}_{\Lambda^{w'w}}$ for which $\Lambda^{w_j} = -1$ for some $j \in I$ cancel among themselves. It is then left to determine the remaining γ_{Λ^w} by the recurrence relation mentioned earlier.

We now show that for some arbitrary weight Λ which has Λ_i , $i \in I$ being non-negative integers that this is the unique weight among $\Lambda^{w'}$, $w' \in \mathcal{W}'$ having this property. Clearly under these assumptions $(\Lambda^{w_i})_i = (\alpha_i^{\vee}, \Lambda^{w_i}) = -\Lambda_i - 2$ is negative for $i \in I$. Since \mathcal{W}' is generated by all w_i , $i \in I$, then we may consider all $w' = w_{i_1} \dots w_{i_n}$, $i_j \in I$ such that $\ell(w') = n$. Denoting by Φ_+' all those positive roots formed from linear combinations of the subset of simple roots α_i , $i \in I$, then we have the result that $\ell(w_i w') = \ell(w') + 1$ if and only if $w'^{-1}(\alpha_i) \in \Phi_+'$.⁵ In this case we have that $(\Lambda^{w_i w'})_i = -(\alpha_i^{\vee}, \Lambda + \rho) - 1$ where $\alpha = w'^{-1}(\alpha_i) \in \Phi_+'$. Since $(\alpha^{\vee}, \Lambda + \rho) \geq 0$ then $(\Lambda^{w_i w'})_i$ must be negative. Hence all $\Lambda^{w'}$, $w' \in \mathcal{W}'$, $w' \neq 1$, have at least one of $\Lambda^{w'_i}$, $i \in I$ which is negative.

If some $\Lambda^{w'w} = \Lambda^{u'u}$ for some $w', u' \in \mathcal{W}'$ and with Λ^{w_i} , $\Lambda^{u_i} \geq 0$, $i \in I$, then clearly we may write $\Lambda^u = \Lambda^{u'^{-1}w'w}$ which contradicts the above unless $u = w$, $u' = w'$. Thus any two sets of such weights $\{\Lambda^{w'w}, w' \in \mathcal{W}'\}$ and $\{\Lambda^{w'u}, w' \in \mathcal{W}'\}$ for $w \neq u$ are disjoint.

⁵ The argument for this is similar to one given in [24].

If for some weight $\Lambda_j = -1$ for some $j \in I$ then $\Lambda^{w_j} = \Lambda$ since $(\alpha_j^\vee, \Lambda^w + \rho) = 0$. We claim that the only $\Lambda^{w'}$ for $w' \in \mathcal{W}'$ which equal Λ in this case are those for which w' is composed of the w_j . For simplicity we consider the case when $\Lambda_j = -1$ and otherwise Λ_i is non-negative for $i \in I$. In this case if $\Lambda^{w'} = \Lambda$ then clearly $\Lambda^{w_j w'} = \Lambda$ so that, for $\alpha = w^{-1}(\alpha_j)$, $(\Lambda^{w_j w'})_j = -(\alpha^\vee, \Lambda + \rho) - 1 = -1$ by assumption. Thus $(\alpha^\vee, \Lambda + \rho) = 0$ which is only the case if $\alpha \propto \alpha_j$. The only roots with this property are $\pm \alpha_i$ so that $w' = 1, w_j$. Also this implies that, for $w', u' \in \mathcal{W}'$, $\Lambda^{w'} = \Lambda^{u'}$ if and only if $u' = w'$ or $u' = w' w_j$. The generalisation is clear.

Appendix B. Weyl character formulae for $SO(d)$

We now consider $SO(d)$ character formulae. We define the variables $x_i = e^{e_i}(\mu) = e^{\mu_i}$ for some arbitrary weight $SO(d)$ $\mu = \sum_{i=1}^r \mu_i e_i$.

For $SO(2r)$ the action of the Weyl group, $\mathcal{W}_{2r} = \mathcal{S}_r \ltimes \mathbb{Z}^{r-1}$, on weights in the orthonormal basis is given by \mathcal{S}_r permutations on the labels followed by reflections involving an even number of sign flips in the labels. This means that for $\varrho = \rho_i \dots \rho_j \in \mathbb{Z}^{r-1}$ where $\rho_i(\ell_1, \dots, \ell_i, \dots, \ell_r) = (\ell_1, \dots, -\ell_i, \dots, \ell_r)$, $\rho_i^2 = 1$ then the number of ρ_i in the composition of ϱ is even and $\text{sgn}(\varrho) = 1$. Using that

$$\sum_{\sigma \in \mathcal{S}_r} \text{sgn}(\sigma) x_{\sigma(1)}^{\ell_1} \dots x_{\sigma(r)}^{\ell_r} = \det[x_i^{\ell_j}], \quad (\text{B.1})$$

and the restriction mentioned on $\varrho \in \mathbb{Z}^{r-1}$ then [25],

$$\mathfrak{W}_{2r} \left(\prod_{i=1}^r x_i^{\ell_i} \right) = \frac{1}{2} \det[x_i^{\ell_j} + x_i^{-\ell_j}] + \frac{1}{2} \det[x_i^{\ell_j} - x_i^{-\ell_j}], \quad (\text{B.2})$$

for \mathfrak{W}_{2r} denoting the $SO(2r)$ Weyl symmetry operator. For some highest weight $\ell = \sum_{i=1}^r \ell_i e_i$ the corresponding Verma module character is given by,

$$\begin{aligned} C_{\underline{\ell}}^{(2r)}(x) &\equiv C_{\underline{\ell}}^{(2r)}(\mu) = \prod_{i=1}^r x_i^{\ell_i} \prod_{1 \leq j < k \leq r, \varepsilon = \pm} (1 - e^{(\alpha_{jk, \varepsilon}, \mu)})^{-1} \\ &= \prod_{i=1}^r x_i^{\ell_i + r - i} \Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1}, \end{aligned} \quad (\text{B.3})$$

where $\alpha_{ij, \pm} = -e_i \pm e_j$, $1 \leq i < j \leq r$ are the negative roots and $\Delta(x)$ is the Vandermonde determinant (4.13). Using the fact that $\Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})$ is left alone by any $\varrho \sigma \in \mathcal{S}_r \ltimes \mathbb{Z}^{r-1}$ then we have quite simply that the Weyl character of the irreducible

representation with dominant integral highest weight ℓ (given by $\chi_{\underline{\ell}}^{(2r)}(x) = \mathfrak{W}_{2r}(C_{\underline{\ell}}^{(2r)}(x))$) reduces to

$$\chi_{\underline{\ell}}^{(2r)}(x) = \frac{1}{2} \left(\det[x_i^{k_j} + x_i^{-k_j}] + \det[x_i^{k_j} - x_i^{-k_j}] \right) \Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1}, \quad (\text{B.4})$$

for $k_i = \ell_i + r - i$. The dimension of the irreducible representation is given by

$$\dim(\mathcal{I}_{\underline{\ell}}^{(2r)}) = 2^{r-1} \prod_{i=1}^r \frac{1}{(2r-2i)!} \prod_{1 \leq i < j \leq r} (\ell_i - \ell_j + j - i)(\ell_i + \ell_j + 2r - i - j). \quad (\text{B.5})$$

For $SO(2r+1)$ the action of the Weyl group, $\mathcal{W}_{2r+1} = \mathcal{S}_r \ltimes \mathbb{Z}^r$, on weights in the orthonormal basis is given by \mathcal{S}_r permutations on the labels followed by reflections involving any number of sign flips in the labels. Using (B.1) we therefore have that [25],

$$\mathfrak{W}_{2r+1} \left(\prod_{i=1}^r x_i^{\ell_i} \right) = \det[x_i^{\ell_j} - x_i^{-\ell_j}], \quad (\text{B.6})$$

for \mathfrak{W}_{2r+1} denoting the $SO(2r+1)$ Weyl symmetry operator. This time for some highest weight $\ell = \sum_{i=1}^r \ell_i e_i$ the corresponding Verma module character is given by,

$$\begin{aligned} C_{\underline{\ell}}^{(2r+1)}(x) &\equiv \mathcal{C}_{\underline{\ell}}^{(2r+1)}(\mu) = \prod_{i=1}^r x_i^{\ell_i} \prod_{1 \leq j < k \leq r, \varepsilon = \pm} (1 - e^{(\alpha_{jk, \varepsilon}, \mu)})^{-1} \prod_{1 \leq l \leq r} (1 - e^{(e_l, \mu)})^{-1} \\ &= \prod_{i=1}^r x_i^{\ell_i + \frac{1}{2} + r - i} \Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1} (x_1^{\frac{1}{2}} - x_1^{-\frac{1}{2}})^{-1} \dots (x_r^{\frac{1}{2}} - x_r^{-\frac{1}{2}})^{-1}, \end{aligned} \quad (\text{B.7})$$

since $-e_i$, $1 \leq i \leq r$ are the weights of the extra negative roots in this case. Using that $\Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1} (x_1^{\frac{1}{2}} - x_1^{-\frac{1}{2}})^{-1} \dots (x_r^{\frac{1}{2}} - x_r^{-\frac{1}{2}})^{-1}$ is left alone by any $\varrho \sigma \in \mathcal{S}_r \ltimes \mathbb{Z}^r$ then the Weyl character of the irreducible representation with dominant integral highest weight ℓ is given by, using (B.6) and (B.7),

$$\begin{aligned} \chi_{\underline{\ell}}^{(2r+1)}(x) &= \det[x_i^{k_j} - x_i^{-k_j}] \Delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})^{-1} (x_1^{\frac{1}{2}} - x_1^{-\frac{1}{2}})^{-1} \dots (x_r^{\frac{1}{2}} - x_r^{-\frac{1}{2}})^{-1}, \end{aligned} \quad (\text{B.8})$$

where $k_i = \ell_i + \frac{1}{2} + r - i$. The dimension of the irreducible representation is given by

$$\begin{aligned} \dim(\mathcal{I}_{\underline{\ell}}^{(2r+1)}) &= \prod_{i=1}^r \frac{1}{(2r+1-2i)!} (2\ell_i + 2r + 1 - 2i) \\ &\quad \times \prod_{1 \leq i < j \leq r} (\ell_i - \ell_j + j - i)(\ell_i + \ell_j + 2r + 1 - i - j). \end{aligned} \quad (\text{B.9})$$

Appendix C. Unitarity bounds

Descendant states have bases, for $p = 0, 1, \dots$,

$$\mathcal{B}^{(p)} = \left\{ \prod_{\substack{v=i\varepsilon, 0 \\ 1 \leq i \leq r, \varepsilon=\pm}} \mathcal{P}_v^{n_v} |\Delta; \underline{\ell}'\rangle, \quad \sum_v n_v = p \right\}, \quad (\text{C.1})$$

for n_v being positive integers (with $n_0 = 0$ for $SO(2r)$) and ℓ' being a weight in the weight system of the module of $SO(d)$ with highest weight ℓ . For $p = 0$ the norms of corresponding states are strictly positive for ℓ' being weights in the $SO(d)$ irreducible representation with dominant integral highest weight ℓ and $\| |\Delta, \ell'\rangle \|^2 = \langle \Delta, \underline{\ell}' | \Delta, \underline{\ell}' \rangle > 0$.

Examining the simplest descendant states with basis $\mathcal{B}^{(1)}$ then these have $SO(d)$ highest weight states

$$\mathcal{H}_d = \{ |\Delta + 1; \underline{\ell} + \underline{v}\rangle \}, \quad (\text{C.2})$$

where $\underline{v} = \varepsilon \mathbf{e}_i$ along with $\underline{v} = 0$ for $d = 2r+1$, these of course occurring in the decomposition of the product between the vector representation and the representation with highest weight ℓ into irreducible representations, $\mathbf{e}_1 \otimes \ell = \bigoplus_v \ell \oplus \underline{v}$. Remarkably, most of the restrictions necessary for the states in \mathcal{H}_d to have positive definite norm are sufficient for the unitarity constraints to be satisfied for all descendant states in $\mathcal{B}^{(p)}$ - also conjectured in [20].

The simplest states in \mathcal{H}_d may be constructed explicitly and are

$$\begin{aligned} |\Delta + 1; \underline{\ell} + \underline{e}_1\rangle &= \mathcal{P}_{1+} |\Delta; \underline{\ell}\rangle^{\text{h.w.}}, \\ |\Delta + 1; \underline{\ell} + \underline{e}_2\rangle &= (-2i(\ell_1 - \ell_2) \mathcal{P}_{2+} + \mathcal{P}_{1+} E_{12}^{-+}) |\Delta; \underline{\ell}\rangle^{\text{h.w.}}, \\ |\Delta + 1; \underline{\ell} + \underline{e}_3\rangle &= (-4(\ell_1 - \ell_3 + 1)(\ell_2 - \ell_3) \mathcal{P}_{3+} - 2i(\ell_1 - \ell_3 + 1) \mathcal{P}_{2+} E_{23}^{-+} \\ &\quad + (\ell_2 - \ell_3 + 1) \mathcal{P}_{1+} E_{12}^{-+} E_{23}^{-+} - (\ell_2 - \ell_3) \mathcal{P}_{1+} E_{23}^{-+} E_{12}^{-+}) |\Delta; \underline{\ell}\rangle^{\text{h.w.}}, \end{aligned} \quad (\text{C.3})$$

which are all annihilated by $SO(d)$ raising operators. Using the conformal algebra and the unitarity conditions, the norms of these three states are given by,

$$\begin{aligned} \| |\Delta + 1; \underline{\ell} + \underline{e}_1\rangle \|^2 &= 4(\Delta + \ell_1) \| |\Delta; \underline{\ell}\rangle \|^2, \\ \| |\Delta + 1; \underline{\ell} + \underline{e}_2\rangle \|^2 &= 16(\Delta + \ell_2 - 1)(\ell_1 - \ell_2)(\ell_1 - \ell_2 + 1) \| |\Delta; \underline{\ell}\rangle \|^2, \\ \| |\Delta + 1; \underline{\ell} + \underline{e}_3\rangle \|^2 &= 64(\Delta + \ell_3 - 2)(\ell_1 - \ell_3 + 1)(\ell_1 - \ell_3 + 2) \\ &\quad \times (\ell_2 - \ell_3)(\ell_2 - \ell_3 + 1) \| |\Delta; \underline{\ell}\rangle \|^2, \end{aligned} \quad (\text{C.4})$$

and for these to be strictly positive this places obvious restrictions on Δ . Other examples which may be readily achieved are, for $SO(3, 2)$,

$$\begin{aligned} |\Delta + 1; \ell + 1\rangle &= \mathcal{P}_{1+} |\Delta; \ell\rangle^{\text{h.w.}}, \quad |\Delta + 1; \ell\rangle = (-2i\ell\mathcal{P}_0 + \mathcal{P}_{1+}E_1^-) |\Delta; \ell\rangle^{\text{h.w.}}, \\ |\Delta + 1; \ell - 1\rangle &= (2\ell(2\ell - 1)\mathcal{P}_{1-} - 2i(2\ell - 1)\mathcal{P}_0E_1^- + \mathcal{P}_{1+}(E_1^-)^2) |\Delta; \ell\rangle^{\text{h.w.}}, \end{aligned} \quad (\text{C.5})$$

for which the norms are

$$\begin{aligned} \||\Delta + 1; \ell + 1\rangle\|^2 &= 4(\Delta + \ell) \||\Delta; \ell\rangle\|^2, \quad \||\Delta + 1; \ell\rangle\|^2 = 8(\Delta - 1)\ell(\ell + 1) \||\Delta; \ell\rangle\|^2 \\ \||\Delta + 1; \ell - 1\rangle\|^2 &= 16(\Delta - \ell - 1)\ell^2(4\ell^2 - 1) \||\Delta; \ell\rangle\|^2. \end{aligned} \quad (\text{C.6})$$

Another important example is for $SO(4, 2)$ whereby along with (C.3), for $\underline{\ell} = (\ell_1, \ell_2)$, we also have

$$\begin{aligned} |\Delta + 1; \ell_1, \ell_2 - 1\rangle &= (-2i(\ell_1 + \ell_2)\mathcal{P}_{2-} + \mathcal{P}_{1+}E_{12}^{--}) |\Delta; \underline{\ell}\rangle^{\text{h.w.}}, \\ |\Delta + 1; \ell_1 - 1, \ell_2\rangle &= (-4(\ell_1^2 - \ell_2^2)\mathcal{P}_{1-} - 2i(\ell_1 - \ell_2)\mathcal{P}_{2+}E_{12}^{--} \\ &\quad - 2i(\ell_1 + \ell_2)\mathcal{P}_{2-}E_{12}^{++} + \mathcal{P}_{1+}E_{12}^{++}E_{12}^{--}) |\Delta; \underline{\ell}\rangle^{\text{h.w.}}, \end{aligned} \quad (\text{C.7})$$

for which the norms are

$$\begin{aligned} \||\Delta + 1; \ell_1, \ell_2 - 1\rangle\|^2 &= 16(\Delta - \ell_2 - 1)(\ell_1 + \ell_2)(\ell_1 + \ell_2 + 1) \||\Delta; \underline{\ell}\rangle\|^2, \\ \||\Delta + 1; \ell_1 - 1, \ell_2\rangle\|^2 &= 64(\Delta - \ell_1 - 2)(\ell_1^2 - \ell_2^2)(\ell_1 - \ell_1 + 1)(\ell_1 + \ell_2 + 1) \||\Delta; \underline{\ell}\rangle\|^2. \end{aligned} \quad (\text{C.8})$$

Constructing other such elements of \mathcal{H}_d is cumbersome. We here outline a simpler procedure for finding the unitarity constraints for $\mathcal{B}^{(1)}$. The norms of the highest weight states in \mathcal{H}_d are more generally given by,

$$\||\Delta + 1; \underline{\ell} + \underline{\mathbf{v}}\rangle\|^2 = \left(\Delta + g_{\underline{\ell}}^{(\mathbf{v})}\right) f_{\underline{\ell}}^{(\mathbf{v})}, \quad (\text{C.9})$$

where the functions $f_{\underline{\ell}}^{(\mathbf{v})}$ are strictly positive for ℓ being strictly inside the dominant Weyl chamber, (3.1) or (3.2). We have that, assuming that ℓ is strictly inside the dominant Weyl chamber,

$$\mathcal{K}_{1-} |\Delta + 1; \underline{\ell} + \underline{\mathbf{v}}\rangle = 0 \quad \Rightarrow \quad \Delta + g_{\underline{\ell}}^{(\mathbf{v})} = 0, \quad (\text{C.10})$$

which is in turn implied by the state $|\Delta + 1; \underline{\ell} + \underline{\mathbf{v}}\rangle$ being null. As an aid to solving (C.10) we extend the definition of $\mathcal{B}^{(1)}$ in (C.1) and consider $\ell' \in \mathcal{V}_{\ell}$ (the Verma module with dominant integral highest weight ℓ). Consider the following highest weight states with respect to $SO(d)$, namely,

$$|\Delta + 1; \underline{\ell}^{w_{\mathbf{v}}} + \underline{\mathbf{e}}_1\rangle = \mathcal{P}_{1+} |\Delta; \underline{\ell}^{w_{\mathbf{v}}}\rangle, \quad \mathcal{K}_{1-} |\Delta + 1; \underline{\ell}^{w_{\mathbf{v}}} + \underline{\mathbf{e}}_1\rangle = 0 \quad \Rightarrow \quad \Delta + \ell_1^{w_{\mathbf{v}}} = 0, \quad (\text{C.11})$$

where w_v are such members of the Weyl group \mathcal{W}_r for which

$$\ell^{w_v} + e_1 = (\ell + w_v^{-1}(e_1))^{w_v} = (\ell + v)^{w_v}, \quad (C.12)$$

for some $v = \varepsilon e_j$ for $\varepsilon = \pm$.⁶ Thus the states (C.11) are related to those in \mathcal{H}_d by the action of $SO(d)$ lowering operators on $|\Delta + 1; \underline{\ell} + \underline{v}\rangle$.⁷ Also, \mathcal{K}_{1-} commutes with all such lowering operators so that the conditions (C.10) and (C.11) should be identical. Thus, using (3.14) (for $\varepsilon_1 = \varepsilon$, $\sigma(1) = j$),

$$g_{\underline{\ell}}^{(\varepsilon e_j)} = \ell_1^{w_{\varepsilon e_j}} = \varepsilon \ell_j + (\varepsilon - 1)\frac{1}{2}d - \varepsilon j + 1, \quad (C.13)$$

determining $g_{\underline{\ell}}^{(v)}$, $v \neq 0$ in (C.9) for the states in \mathcal{H}_d .

To deal with the state $|\Delta + 1; \ell\rangle \in \mathcal{H}_{2r+1}$ for $SO(2r+1, 2)$ we first note an interesting observation. Consider the state $|\Delta + 1; \underline{\ell} + \underline{e}_{r+1}\rangle \in \mathcal{H}_{2r+2}$ which is given by

$$\begin{aligned} & |\Delta + 1; \underline{\ell} + \underline{e}_{r+1}\rangle \\ &= \left(A_{\underline{\ell}} \mathcal{P}_{r+1} + \sum_{1 \leq i \leq r, \sigma} B_{\underline{\ell}, i, \sigma} \mathcal{P}_i E_{\sigma(i i+1)}^{-+} E_{\sigma(i+1 i+2)}^{-+} \cdots E_{\sigma(r r+1)}^{-+} \right) |\Delta, \underline{\ell}\rangle^{\text{h.w.}}, \end{aligned} \quad (C.14)$$

where σ permutes $(i i+1), \dots, (r r+1)$ and $A_{\underline{\ell}}, B_{\underline{\ell}, i, \sigma}$ are determined from the requirement that $E_{12}^{+-}, \dots, E_{r r+1}^{+-}$ annihilate the state ($E_{r r+1}^{++}$ automatically annihilates it). Defining $\tilde{A}_{(\ell_1, \dots, \ell_r)} = A_{(\ell_1, \dots, \ell_r, 0)}$, $\tilde{B}_{(\ell_1, \dots, \ell_r), i, \sigma} = B_{(\ell_1, \dots, \ell_r, 0), i, \sigma}$, then we claim that $|\Delta + 1; \ell\rangle \in \mathcal{H}_{2r+1}$ is given by

$$|\Delta + 1; \ell\rangle = \left(\tilde{A}_{\underline{\ell}} \mathcal{P}_0 + \sum_{1 \leq i \leq r, \sigma} \tilde{B}_{\underline{\ell}, i, \sigma} \mathcal{P}_i^+ E_{\sigma(i i+1)}^{-+} E_{\sigma(i+1 i+2)}^{-+} \cdots E_{\sigma(r)}^{-+} \right) |\Delta, \underline{\ell}\rangle^{\text{h.w.}}, \quad (C.15)$$

where now σ permutes $(i i+1), \dots, (r)$. This follows when we show that the conditions on $\tilde{A}_{\underline{\ell}}, \tilde{B}_{\underline{\ell}, i, \sigma}$ arising from $E_{i i+1}^{+-}, E_r^+, 1 \leq i \leq r-1$ annihilating (C.15) are exactly equivalent to those on $A_{\underline{\ell}}, B_{\underline{\ell}, i, \sigma}$ arising from $E_{i i+1}^{+-}, E_{r r+1}^{+-}, 1 \leq i \leq r-1$ annihilating (C.14) for $\ell_{r+1} = 0$ if we identify \mathcal{P}_0 with \mathcal{P}_{r+1} and E_r^- with $E_{r r+1}^{-+}$. We have that $[\mathcal{K}_{1-}, \mathcal{P}_{r+1}] = -2i E_{1 r+1}^{-+} = (-2i)^{1-r} [E_{12}^{-+}, \dots, [E_{r-1 r}, E_{r r+1}^{-+}] \dots]$ and $[\mathcal{K}_{1-}, \mathcal{P}_0] = -2i E_1^- = (-2i)^{1-r} [E_{12}^{-+}, \dots, [E_{r-1 r}, E_r^-] \dots]$. Due to this and as $\tilde{A}_{(\ell_1, \dots, \ell_r)} = A_{(\ell_1, \dots, \ell_r, 0)}$, $\tilde{B}_{(\ell_1, \dots, \ell_r), i, \sigma} = B_{(\ell_1, \dots, \ell_r, 0), i, \sigma}$ then \mathcal{K}_{1-} annihilating (C.14) for $\ell_{r+1} = 0$ results in the same equations for Δ as for \mathcal{K}_{1-} annihilating (C.15) if we identify $E_{r r+1}^{-+}$ with E_r^- . Thus, from (C.13) for $j = r+1$, $\varepsilon = +$, $\ell_{r+1} = 0$,

$$\mathcal{K}_{1-} |\Delta + 1, \underline{\ell}\rangle = 0 \quad \Rightarrow \quad g_{\underline{\ell}}^{(0)} = -r. \quad (C.16)$$

⁶ Note that for $SO(2r+1)$ the vector $v = 0$ is not on the Weyl orbit of e_1 - we need a different approach to deal with this.

⁷ For instance, we have that $|\Delta + 1; \underline{\ell}^{\sigma_{12}} + \underline{e}_1\rangle = (E_{12}^{-+})^{\ell_1 - \ell_2} |\Delta + 1; \underline{\ell} + \underline{e}_2\rangle$ for $\sigma_{12}(\ell_1, \ell_2, \dots) = (\ell_2, \ell_1, \dots)$ and with $|\Delta + 1, \underline{\ell} + \underline{e}_2\rangle$ given in (C.7).

Now that we have determined $g_{\underline{\ell}}^{(\mathbf{v})}$ in (C.9) to be given by (C.13) and (C.16), we may determine the unitarity bounds for states in $\mathcal{B}^{(1)}$, the simplest descendants. For \mathcal{H}_d and $\ell_1 = \dots = \ell_p > |\ell_{p+1}|$, $p \leq r-1$ then we have that $f_{\underline{\ell}}^{(\mathbf{v})} = 0$ in (C.9) for $\mathbf{v} = -\mathbf{e}_1, \varepsilon \mathbf{e}_j, \mathbf{e}_p$, $j = 2, \dots, p-1$ ⁸ and that

$$\begin{aligned} \Delta &\geq \max.\{-g_{\underline{\ell}}^{(\mathbf{e}_1)}, -g_{\underline{\ell}}^{(-\mathbf{e}_p)}, -g_{\underline{\ell}}^{(\mathbf{e}_j)}, -g_{\underline{\ell}}^{(-\mathbf{e}_j)}, p+1 \leq j \leq r\} \cup \{-g_{\underline{\ell}}^{(0)} \text{ for } d = 2r+1\} \\ &= -g_{\underline{\ell}}^{(-\mathbf{e}_p)} = \ell_1 + d - p - 1, \end{aligned} \quad (\text{C.17})$$

which matches the first requirement in (3.8). At the unitarity bound $\Delta = \ell_1 + d - p - 1$ then all the states $|\Delta + 1; \underline{\ell} + \underline{\mathbf{v}}\rangle$ for $\mathbf{v} = -\mathbf{e}_1, \varepsilon \mathbf{e}_j$, $2 \leq j \leq p$ in \mathcal{H}_d are null.

In even dimensions, for $\ell_1 = \dots = \pm \ell_r$ for \mathcal{H}_{2r} then we have that $f_{\underline{\ell}}^{(\mathbf{v})} = 0$ in (C.9) for $\mathbf{v} = -\mathbf{e}_1, \varepsilon \mathbf{e}_j, \pm \mathbf{e}_r$, $j = 2, \dots, r-1$

$$\Delta \geq \max.\{-g_{\underline{\ell}}^{(\mathbf{e}_1)}, -g_{\underline{\ell}}^{(\mp \mathbf{e}_r)}\} = \ell_1 + r - 1, \quad (\text{C.18})$$

with in addition the state $|\Delta + 1; \underline{\ell} \mp \underline{\mathbf{e}}_r\rangle$ being null at the unitarity bound.

In odd dimensions, for $\ell_1 = \dots = \ell_r > \frac{1}{2}$ for \mathcal{H}_{2r+1} then $f_{\underline{\ell}}^{(\mathbf{v})} = 0$ in (C.9) for $\mathbf{v} = -\mathbf{e}_1, \varepsilon \mathbf{e}_j, \mathbf{e}_r$, $j = 2, \dots, r-1$ and

$$\Delta \geq \max.\{-g_{\underline{\ell}}^{(\mathbf{e}_1)}, -g_{\underline{\ell}}^{(-\mathbf{e}_r)}, -g_{\underline{\ell}}^{(0)}\} = \ell_1 + r, \quad (\text{C.19})$$

with in addition the state $|\Delta + 1, \underline{\ell} - \underline{\mathbf{e}}_r\rangle$ being null at the unitarity bound. For $\ell_1 = \dots = \ell_r = \frac{1}{2}$ then $f_{\underline{\ell}}^{(\mathbf{v})} = 0$ in (C.9) for $\mathbf{v} = -\mathbf{e}_1, \varepsilon \mathbf{e}_j$, $j = 2, \dots, r$ and

$$\Delta \geq \max.\{-g_{\underline{\ell}}^{(\mathbf{e}_1)}, -g_{\underline{\ell}}^{(0)}\} = r = [\frac{1}{2}d], \quad (\text{C.20})$$

with the state $|\Delta + 1, \underline{\ell}\rangle$ being null at the unitarity bound.

Appendix D. Expansion and product formulae

In this appendix various formulae from section four are proven. We make use of a simple property of the function $P^{(d)}(s, \mathbf{x})$ defined for $d = 2r$ in (3.19) and $d = 2r+1$ in (3.30). Under the action of the Weyl symmetry operator \mathfrak{W}_d (defined in appendices A,B) it obeys

$$\mathfrak{W}_d(f(s, \mathbf{x})P^{(d)}(s, \mathbf{x})) = \mathfrak{W}_d(f(s, \mathbf{x})) P^{(d)}(s, \mathbf{x}), \quad (\text{D.1})$$

⁸ Here and in the following, this is because the corresponding states $|\Delta + 1; \underline{\ell} + \underline{\mathbf{v}}\rangle$ for ℓ being on the boundary of the dominant Weyl chamber are null as $SO(d)$ representations.

for any $f(s, \mathbf{x})$, as $P^{(d)}(s, \mathbf{x})$ is invariant under the action of any element of the $SO(d)$ Weyl group, \mathcal{W}_d . Note also that \mathcal{W}_d has no effect on the variable s .

We discuss the even dimensional cases of (4.11), (4.14), (4.15) first. For (4.14) we have that,

$$\begin{aligned} \sum_{q=0}^{\infty} s^{\ell+r+q-1} \chi_{(\ell+q, \ell, \dots, \pm \ell)}^{(2r)}(\mathbf{x}) &= s^{\ell+r-1} \mathfrak{W}_{2r} \left(\sum_{q=0}^{\infty} (s x_1)^q C_{(\ell, \dots, \pm \ell)}^{(2r)}(\mathbf{x}) \right) \\ &= s^{\ell+r-1} \mathfrak{W}_{2r} \left(\frac{1}{1 - s x_1} C_{(\ell, \dots, \pm \ell)}^{(2r)}(\mathbf{x}) \right), \end{aligned} \quad (\text{D.2})$$

which follows just by the definition of the character of the irreducible representation (B.4) in terms of the Verma module character (B.3). Using (D.1) then (D.2) may be rewritten as

$$\begin{aligned} s^{\ell+r-1} P^{(2r)}(s, \mathbf{x}) \mathfrak{W}_{2r} \left((1 - s x_1)^{-1} \prod_{i=2}^r (1 - s x_i) (1 - s x_i^{-1}) C_{(\ell, \dots, \pm \ell)}^{(2r)}(\mathbf{x}) \right) \\ = s^{\ell+r-1} P^{(2r)}(s, \mathbf{x}) \sum_{\substack{n_{i-}, n_{j+}=0,1 \\ 0 \leq n = \sum n_{i\epsilon} \leq 2r-2}} (-s)^n \chi_{(\ell-n_{1-}, \ell+n_{2+}-n_{2-}, \dots, \pm \ell+n_{r+}-n_{r-})}^{(2r)}(\mathbf{x}). \end{aligned} \quad (\text{D.3})$$

For $n > 0$ in (D.3) we may use

$$\chi_{(\ell_1, \dots, \ell_j, \ell-1, \ell+1, \ell_{j+3}, \dots, \ell_r)}^{(d)}(\mathbf{x}) = -\chi_{(\ell_1, \dots, \ell_j, \ell, \ell, \ell_{j+3}, \dots, \ell_r)}^{(d)}(\mathbf{x}), \quad (\text{D.4})$$

to show that the contributions for given n reduce to a single one from

$$\chi_{(\ell, \dots, \ell, \ell-1, \dots, \ell-1, \pm \ell \mp 1)}^{(2r)}(\mathbf{x}), \quad (\text{D.5})$$

\uparrow
 n^{th} position

with all contributions for $n > r$ vanishing. Hence we have that (D.2) reduces to $\mathcal{D}_{[\ell+r-1, \ell] \pm}^{(2r)}(s, \mathbf{x})$ defined in (3.26) thus proving (4.14). Notice that (4.11) is a special case of (4.14) when we take $\ell = 0$ in the latter (whereby, as mentioned before, $\mathcal{D}_{[\ell+r-1, \ell] \pm}^{(2r)}(s, \mathbf{x}) \rightarrow s^{r-1} (1 - s^2) P^{(2r)}(s, \mathbf{x})$).

We may prove (4.15) in a very similar way. The sum on the right hand side of (4.15) may be reduced to

$$s^{\ell+r+j-1} \mathfrak{W}_{2r} \left((1 - s x_1)^{-1} \prod_{i=r-j+1}^r (1 - s x_i)^{-1} (1 - s x_i^{-1})^{-1} C_{(\ell, \dots, \ell, \ell_1, \dots, \ell_j)}^{(2r)}(\mathbf{x}) \right), \quad (\text{D.6})$$

in a similar fashion as (D.2), when we perform the sums over p_1, \dots, p_j, q . This may be rewritten using (D.1) as

$$\begin{aligned}
& s^{\ell+r+j-1} P^{(2r)}(s, \mathbf{x}) \mathfrak{W}_{2r} \left((1 - sx_1) \prod_{i=2}^{r-j} (1 - sx_i) (1 - sx_i^{-1}) C_{(\ell, \dots, \ell, \ell_1, \dots, \ell_j)}^{(2r)}(\mathbf{x}) \right) \\
&= s^{\ell+r+j-1} P^{(2r)}(s, \mathbf{x}) \sum_{\substack{n_{i-}, n_{j+}=0,1 \\ 0 \leq n = \sum n_{i\epsilon} \leq 2r-2j-2}} (-s)^n \chi_{(\ell-n_{1-}, \ell+n_{2+}-n_{2-}, \dots, \ell+n_{r-j+}-n_{r-j-}, \ell_1, \dots, \ell_r)}^{(2r)}(\mathbf{x}).
\end{aligned} \tag{D.7}$$

For similar reasons as before, for $n > 0$ the contributions for given n reduce to a single one from

$$\chi_{(\ell, \dots, \ell, \ell-1, \dots, \ell-1, \ell_1, \dots, \ell_j)}^{(2r)}(\mathbf{x}), \tag{D.8}$$

\uparrow
 $n^{\text{th}} \text{ position}$

so that (D.7) equals $\mathcal{D}_{[\ell+r+j-1, \ell, \ell_1, \dots, \ell_j]}^{(2r)}(s, \mathbf{x})$ in (3.25) for $p = r - j$.

Turning to the odd dimensional cases of (4.27), (4.29), (4.30), (4.31) these may be proven in a very similar way as for the even dimensional cases when we use (D.1). Note that we may use the definition of the irreducible character (B.8) in terms of the Verma module character (B.7) and (3.30) to rewrite the sum on the right hand side of (4.30) as

$$\begin{aligned}
& s^r P^{(2r+1)}(s, \mathbf{x}) (1 - s) \mathfrak{W}_{2r+1} \left((1 - sx_1^{-1}) \prod_{i=2}^r (1 - sx_i) (1 - sx_i^{-1}) C_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(2r+1)}(\mathbf{x}) \right) \\
&= s^r P^{(2r+1)}(s, \mathbf{x}) (1 - s) \chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(2r+1)}(\mathbf{x}),
\end{aligned} \tag{D.9}$$

which matches (3.34). The free scalar case of (4.29) follows in a similar fashion. The identity (4.27) is in fact equivalent to (4.29). The sum on the right hand side of (4.31) may be rewritten as

$$s^{\ell+r+j} \mathfrak{W}_{2r+1} \left((1 - s)^{-1} (1 - sx_1)^{-1} \prod_{i=r-j+1}^r (1 - sx_i)^{-1} (1 - sx_i^{-1})^{-1} C_{(\ell, \dots, \ell, \ell_1, \dots, \ell_j)}^{(2r+1)}(\mathbf{x}) \right), \tag{D.10}$$

when we perform the sums over p_i, q, t . This may be rewritten as

$$\begin{aligned}
& s^{\ell+r+j} P^{(2r+1)}(s, \mathbf{x}) \mathfrak{W}_{2r+1} \left((1 - sx_1) \prod_{i=2}^{r-j} (1 - sx_i) (1 - sx_i^{-1}) C_{(\ell, \dots, \ell, \ell_1, \dots, \ell_j)}^{(2r+1)}(\mathbf{x}) \right) \\
&= s^{\ell+r+j} P^{(2r+1)}(s, \mathbf{x}) \sum_{\substack{n_{i-}, n_{j+}=0,1 \\ 0 \leq n = \sum n_{i\epsilon} \leq 2r-2j-2}} (-s)^n \chi_{(\ell-n_{1-}, \ell+n_{2+}-n_{2-}, \dots, \ell+n_{r-j+}-n_{r-j-}, \ell_1, \dots, \ell_r)}^{(2r+1)}(\mathbf{x}).
\end{aligned} \tag{D.11}$$

Using (D.4), for $n > 0$ the contributions for given n reduce to a single one from

$$\chi_{(\ell, \dots, \ell, \ell-1, \dots, \ell-1, \ell_1, \dots, \ell_j)}^{(2r+1)}(\mathbf{x}), \quad (D.12)$$

\uparrow
 $n^{\text{th}} \text{ position}$

so that (D.11) equals $\mathcal{D}_{[\ell+r+j, \ell, \ell_1, \dots, \ell_j]}^{(2r+1)}(s, \mathbf{x})$ in (3.32) for $p = r - j$.

To prove some product formulae we will use the expansions above and the following,

$$\mathfrak{W}_d(f(\mathbf{x})\chi_{\underline{\ell}}^{(d)}(\mathbf{x})) = \mathfrak{W}_d(f(\mathbf{x}))\chi_{\underline{\ell}}^{(d)}(\mathbf{x}), \quad (D.13)$$

for any $f(\mathbf{x})$.

For even dimensions we now prove (4.22) and (4.23) for $\ell' = \frac{1}{2}$. In this case we have that

$$\mathcal{D}_{[\frac{1}{2}+2m; \frac{1}{2}] \pm}^{(4m+2)}(s, \mathbf{x}) = s^{\frac{1}{2}+2m} \left(\chi_{(\frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})}^{(4m+2)}(\mathbf{x}) - s \chi_{(\frac{1}{2}, \dots, \frac{1}{2}, \mp \frac{1}{2})}^{(4m+2)}(\mathbf{x}) \right), \quad (D.14)$$

where we may determine from (B.4) that,

$$\begin{aligned} \chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(4m+2)}(\mathbf{x}) &= \prod_{i=1}^{2m+1} x_i^{-\frac{1}{2}} \sum_{\substack{m \geq t \geq 0 \\ 2m+1 \geq j_1 > \dots > j_{2t+1} \geq 1}} x_{j_1} \cdots x_{j_{2t+1}}, \\ \chi_{(\frac{1}{2}, \dots, -\frac{1}{2})}^{(4m+2)}(\mathbf{x}) &= \prod_{i=1}^{2m+1} x_i^{-\frac{1}{2}} \left(1 + \sum_{\substack{m \geq t \geq 1 \\ 2m+1 \geq j_1 > \dots > j_{2t} \geq 1}} x_{j_1} \cdots x_{j_{2t}} \right). \end{aligned} \quad (D.15)$$

We use (D.14) and (4.14) to expand $\mathcal{D}_{[\ell+2m, \ell] \pm}^{(4m+2)}(s, \mathbf{x})$ in (4.22) and then match powers of s on both sides. Clearly the $O(1)$ terms on both sides of (4.22) agree. At $O(s^q)$ for $q \geq 1$ we must show that,

$$\begin{aligned} &\chi_{(\ell+q, \ell, \dots, \ell)}^{(4m+2)}(\mathbf{x})\chi_{(\frac{1}{2}, \dots, -\frac{1}{2})}^{(4m+2)}(\mathbf{x}) - \chi_{(\ell+q-1, \ell, \dots, \ell)}^{(4m+2)}(\mathbf{x})\chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(4m+2)}(\mathbf{x}) \\ &= \sum_{\substack{t_i = \ell \pm \frac{1}{2} \\ t_i \geq t_{i+1}}} \chi_{(\ell+\frac{1}{2}+q, \ell+\frac{1}{2}, t_1, t_1, \dots, t_{m-1}, t_{m-1}, \ell-\frac{1}{2})}^{(4m+2)}(\mathbf{x}) - \chi_{(\ell+\frac{1}{2}+q-2, \ell+\frac{1}{2}, t_1, t_1, \dots, t_{m-1}, t_{m-1}, \ell-\frac{1}{2})}^{(4m+2)}(\mathbf{x}). \end{aligned} \quad (D.16)$$

Using (D.13) and (D.15) we may rewrite the left hand side of (D.16) as

$$\begin{aligned} &\mathfrak{W}_{4m+2} \left(C_{(\ell+q, \ell, \dots, \ell)}^{(4m+2)}(\mathbf{x})\chi_{(\frac{1}{2}, \dots, -\frac{1}{2})}^{(4m+2)}(\mathbf{x}) - C_{(\ell+q-1, \ell, \dots, \ell)}^{(4m+2)}(\mathbf{x})\chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(4m+2)}(\mathbf{x}) \right) \\ &= \mathfrak{W}_{4m+2} \left(C_{(\ell+q-\frac{3}{2}, \ell-\frac{1}{2}, \dots, \ell-\frac{1}{2})}^{(4m+2)}(\mathbf{x})(x_1^2 - 1) \sum_{\substack{m \geq t \geq 1 \\ 2m+1 \geq j_1 > \dots > j_{2t-1} \geq 2}} x_{j_1} \cdots x_{j_{2t-1}} \right). \end{aligned} \quad (D.17)$$

For $q \geq 1$ most of the terms in (D.17) vanish under the action of the Weyl symmetry operator and it reduces to

$$\mathfrak{W}_{4m+2} \left(C_{(\ell+q-\frac{3}{2}, \ell-\frac{1}{2}, \dots, \ell-\frac{1}{2})}^{(4m+2)}(x) (x_1^2 - 1) \sum_{t=1}^m x_2 x_3 \cdots x_{2t} \right), \quad (\text{D.18})$$

and from here it is easy to show that this agrees with the right hand side of (D.16).

Similarly, using (D.14) and (4.14) to expand $\mathcal{D}_{[\ell+2m, \ell] \pm}^{(4m+2)}(s, x)$ in (4.23), then matching powers of s on both sides of the equation (4.23) we must show that for $q \geq 0$,

$$\begin{aligned} & \chi_{(\ell+q, \ell, \dots, \varepsilon \ell)}^{(4m+2)}(x) \chi_{(\frac{1}{2}, \dots, \varepsilon \frac{1}{2})}^{(4m+2)}(x) - \chi_{(\ell+q-1, \ell, \dots, \varepsilon \ell)}^{(4m+2)}(x) \chi_{(\frac{1}{2}, \dots, -\varepsilon \frac{1}{2})}^{(4m+2)}(x) \\ &= \sum_{\substack{t_i = \ell \pm \frac{1}{2} \\ t_i \geq t_{i+1}}} \chi_{(\ell + \frac{1}{2} + q, t_1, t_1, \dots, t_m, \varepsilon t_m)}^{(4m+2)}(x) - \chi_{(\ell + \frac{1}{2} + q - 2, t_1, t_1, \dots, t_m, \varepsilon t_m)}^{(4m+2)}(x), \end{aligned} \quad (\text{D.19})$$

for $\varepsilon = \pm$. Using (D.13) and (D.15) we may rewrite the left hand side of (D.19) for $\varepsilon = +$ as

$$\begin{aligned} & \mathfrak{W}_{4m+2} \left(C_{(\ell+q, \ell, \dots, \ell)}^{(4m+2)}(x) \chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(4m+2)}(x) - C_{(\ell+q-1, \ell, \dots, \ell)}^{(4m+2)}(x) \chi_{(\frac{1}{2}, \dots, -\frac{1}{2})}^{(4m+2)}(x) \right) \\ &= \mathfrak{W}_{4m+2} \left(C_{(\ell+q-\frac{3}{2}, \ell-\frac{1}{2}, \dots, \ell-\frac{1}{2})}^{(4m+2)}(x) (x_1^2 - 1) \left(1 + \sum_{\substack{m \geq t \geq 1 \\ 2m+1 \geq j_1 > \dots > j_{2t} \geq 2}} x_{j_1} \cdots x_{j_{2t}} \right) \right). \end{aligned} \quad (\text{D.20})$$

For $q \geq 0$ most of the terms in (D.20) vanish under the action of the Weyl symmetry operator and it reduces to

$$\mathfrak{W}_{4m+2} \left(C_{(\ell+q-\frac{3}{2}, \ell-\frac{1}{2}, \dots, \ell-\frac{1}{2})}^{(4m+2)}(x) (x_1^2 - 1) \left(1 + \sum_{t=1}^m x_2 x_3 \cdots x_{2t+1} \right) \right), \quad (\text{D.21})$$

and from here it is easy to show that this agrees with the right hand side of (D.19) for $\varepsilon = +$.

We also have that

$$\mathcal{D}_{[\frac{1}{2}+2m-1; \frac{1}{2}] \pm}^{(4m)}(s, x) = s^{\frac{1}{2}+2m-1} \left(\chi_{(\frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})}^{(4m)}(x) - s \chi_{(\frac{1}{2}, \dots, \frac{1}{2}, \mp \frac{1}{2})}^{(4m)}(x) \right), \quad (\text{D.22})$$

where,

$$\begin{aligned} \chi_{(\frac{1}{2}, \dots, -\frac{1}{2})}^{(4m)}(x) &= \prod_{i=1}^{2m} x_i^{-\frac{1}{2}} \sum_{\substack{m-1 \geq t \geq 0 \\ 2m \geq j_1 > \dots > j_{2t+1} \geq 1}} x_{j_1} \cdots x_{j_{2t+1}}, \\ \chi_{(\frac{1}{2}, \dots, \frac{1}{2})}^{(4m)}(x) &= \prod_{i=1}^{2m} x_i^{-\frac{1}{2}} \left(1 + \sum_{\substack{m \geq t \geq 1 \\ 2m \geq j_1 > \dots > j_{2t} \geq 1}} x_{j_1} \cdots x_{j_{2t}} \right), \end{aligned} \quad (\text{D.23})$$

and this allows similar product formulae in $d = 4m$ dimensions to be derived straightforwardly in an analogous fashion as above.

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